

TOPICS IN PERCOLATION AND SEQUENCE ANALYSIS

A Thesis
Presented to
The Academic Faculty

by

Chen Xu

In Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy in the
School of Mathematics

Georgia Institute of Technology
August 2018

Copyright © 2018 by Chen Xu

TOPICS IN PERCOLATION AND SEQUENCE ANALYSIS

Approved by:

Professor Christian Houdré,
Committee Chair
School of Mathematics
Georgia Institute of Technology

Professor Christian Houdré, Advisor
School of Mathematics
Georgia Institute of Technology

Professor Robert D. Foley, Co-Advisor
H. Milton Stewart School of Industrial
and System Engineering
Georgia Institute of Technology

Professor Michael Damron
School of Mathematics
Georgia Institute of Technology

Professor Jack Hanson
Department of Mathematics
City College of New York

Professor Gerandy Brito
School of Mathematics
Georgia Institute of Technology

Date Approved: June 22, 2018

To my family

ACKNOWLEDGEMENTS

This thesis could not have been successfully completed without the invaluable assistance of many individuals and institutes. The following list of acknowledgements is, by no means, exhaustive.

I would first like to thank my advisors, Dr. Christian Houdré and Dr. Robert D. Foley for their support and guidance throughout my Ph.D life at Georgia Tech. Not only did they inspire and advise my research, they also helped me navigate my life and career. I will always be grateful for their patient guidance on every aspect of my work and extra miles they went to help transferring me into the Ph.D program in mathematics.

Next, I would like to thank Dr. Michael Damron, Dr. Jack Hanson and Dr. Brito for serving as my committee members. I would like to extend my sincere gratitude to Dr. Michael Damron for his numerous guidance and suggestion on my research in the directed last passage percolation model. I would also like to specially recognize Dr. Antonius Dieker, Dr. Robert D. Foley, Dr. David A. Goldberg, Dr. Christian Houdré and Dr. Vladimir I. Koltchinskii for their extraordinary teaching of probability and statistics related topics, which inspired me to do research in this field.

I am also very much indebted to the School of Mathematics at Georgia Tech for providing me continuous financial support and other resources and opportunities throughout my Ph.D studies. In particular, I would like to extend my gratitude to Dr. John Etnyre, who made my transfer to the program and the follow-up study and research in the School smooth and to Dr. Burke Moreg, who gave one of the best English and American culture training programs to help me with my study, teaching and career development. In addition, my thanks should also go to H. Milton Stewart School of Industrial and System Engineering at Georgia Tech for supporting me for the first two years in my graduate education and exposing me to the field of industrial and applied mathematics and operations research.

There are many of my fellow undergraduate and graduate students whose friendship

has meant so much to me over years. I especially want to thank Dr. Chengliang Zhang for introducing me to the opportunity at Georgia Tech, navigating me through the application and helping me settle down when I first came here. I have shared with great happiness and my friendship with all of them, including Dr. Qiushi Chen, Dr. Weijun Ding, Juntao Duan, Dr. Ruoting Gong, George Kerchev, Changong Li, Junwei Li, Sergio Mayorga, Xinyu Min, Dr. Yanni Ping, Junqing Wang, Dr. Qianyi Wang, Xin Wang, Yuze Zhang and Fan Zhou.

I would also like to take this opportunity to thank all of my mathematics teachers during my early years. Without their education, inspiration and encouragement, I would never have pursued this path thus far. Many of them are worth-mentioning, including Wenjie Ni, Zhenming Shen, Meijuan Yang, Jianxin Zhou, Dr. Jungong Xue and Dr. Weiguo Gao.

Last but absolutely not least, I want to thank my parents, Jiapeng Xu and Yun Wu for their unconditional love and support throughout my life. Without their encouragement and support, I would have not been able to start this journey at all.

Contents

DEDICATION	iii
ACKNOWLEDGEMENTS	iv
LIST OF TABLES	vii
LIST OF FIGURES	viii
SUMMARY	ix
I CONCENTRATION OF GEODESICS IN DIRECTED BERNOULLI PERCOLATION	1
1.1 Introduction	1
1.2 Preliminaries and Main Results	3
1.3 Proof of Main Results	7
1.3.1 Blocks, Decompositions and Concentration	7
1.3.2 Proof of the Main Result	16
1.4 Concluding Remarks	18
II POWER LOWER BOUNDS FOR THE CENTRAL MOMENTS OF THE LAST PASSAGE TIME FOR DIRECTED PERCOLATION IN A THIN RECTANGLE	22
2.1 Introduction and Statements of Results	22
2.2 Preliminaries	24
2.2.1 Linear Growth of M_n	24
2.2.2 Local Reversed Lipschitz Condition	28
2.3 Proof of Theorem 2.1.1	33
2.4 Concluding Remarks	36
III A NOTE ON THE EXPECTED LENGTH OF THE LONGEST COMMON SUBSEQUENCES OF TWO I.I.D. RANDOM PERMUTATIONS	39
3.1 Introduction	39
3.2 Main Results	40
3.3 Further Properties of $L^{(n)}$	44
3.4 Concluding Remarks	47
Bibliography	49

List of Tables

- | | | |
|---|--|----|
| 1 | Numerical evidence shows that both the smallest and the second largest eigenvalues grow at a factorial-like speed. | 46 |
|---|--|----|

List of Figures

1	We study the maximum of the passage times along two particular paths, i.e., two paths going along the upper and the lower edges of the 1×1 blocks on the main diagonal.	4
2	With high probability, the geodesics deviate from the main diagonal by an amount at most of order $\Theta(n^{1-\beta} + n^{1+\kappa\beta-\alpha/2}\sqrt{\ln n} + n^\alpha)$	19

SUMMARY

This thesis studies three topics, two in percolation system and one in sequence analysis. In the first part, we prove that, for directed Bernoulli last passage percolation with i.i.d. weights on vertices over a $n \times n$ grid and for n large enough, the geodesics are shown to be concentrated in a cylinder, centered on the main diagonal and of width of order $n^{(2\kappa+2)/(2\kappa+3)}\sqrt{\ln n}$, where $1 \leq \kappa < \infty$ is the curvature power-index of the shape function at $(1, 1)$. The methodology of proof is robust enough to also apply to directed Bernoulli first passage site percolation, and further to longest common subsequences in random words.

In the second part, we prove that, in directed last passage site percolation over a $n \times \lfloor n^\alpha \rfloor$ -grid and for i.i.d. random weights having finite support, the order of the r -th central moment, $1 \leq r < +\infty$, of the last passage time is, for n large enough, lower bounded by $n^{r(1-\alpha)/2}$, $0 < \alpha < 1/3$.

In the last part, we address a question and a conjecture on the expected length of the longest common subsequences of two i.i.d. random permutations of $[n] := \{1, 2, \dots, n\}$. The question is resolved by showing that the minimal expectation is not attained in the uniform case. The conjecture asserts that \sqrt{n} is a lower bound on this expectation, but we only obtain $\sqrt[3]{n}$ for it.

Chapter I

CONCENTRATION OF GEODESICS IN DIRECTED BERNOULLI PERCOLATION

1.1 Introduction

It has been initially conjectured in [26] that many percolation systems including undirected/directed, first/last passage percolation falls into the KPZ universality class. They are expected to satisfy the scaling relation: $\chi = 2\xi - 1$, where χ and ξ are respectively the shape and the transversal fluctuations exponents. Moreover, χ can also be viewed as the asymptotic order of the standard deviation of the first/last passage time, while geodesics are expected to be confined to a cylinder around the diagonal of width of asymptotic order n^ξ . Specifically, on a two dimensional $n \times n$ grid, it is conjectured that $\chi = 1/3$ and $\xi = 2/3$. However, to date, it has only been shown that $\chi \leq 1/2$ and that $\xi \leq 3/4$, under various types of assumptions. The upper bound $3/4$ is obtained in [30] by showing that $2\xi \leq 1 + \chi'$, where χ' is an exponent closely related to χ and is itself upper-bounded by $1/2$. The relation $2\xi = 1 + \chi$ has also been recently proved under different definitions of ξ and χ in [11] (see also [2]). As for the bounds for the shape fluctuations exponent χ , fewer results are available. To date, a sublinear order $O(\sqrt{n/\ln n})$, in the context of first passage percolation (FPP) with various types of weight distributions has been shown in [7, 13]. For a list of other definitions and results on these topics, we refer the interested readers to the recent comprehensive survey [3].

Transversal fluctuations have also been studied in a related problem, i.e., the analysis of the longest common subsequences (LCSs) in two random words of length n . As well known, LCSs can be viewed as directed last passages in a two-dimensional percolation grid with dependent Bernoulli weights. It is proved in [19] that the optimal alignments corresponding to the LCSs also stay, with high probability, in a sector close to the diagonal. Moreover, when it comes to the shape fluctuation, i.e., the standard deviation of LC_n , the length of

the LCSs, the results are more complete: First, by the Efron-Stein inequality, the shape fluctuations are upper bounded by \sqrt{n} , for arbitrary distributions on any finite dictionary. Second, a lower bound of order \sqrt{n} has been obtained under various asymmetry assumptions ([27, 18, ...]). More noticeably, a central limit theorem has been proved for LC_n in [16].

In this chapter, we mainly study the transversal fluctuations in directed last passage percolation (DLPP) and briefly extend it to other settings. Our methodology shows that, with high probability, geodesics in DLPP are confined to a cylinder, around the main diagonal, of width of order $n^{(2\kappa+2)/(2\kappa+3)}\sqrt{\ln n}$, where $1 \leq \kappa < +\infty$ is the curvature power of the shape function at $(1, 1)$.

The model under study is the classical one: DLPP on a $n \times n$ grid with $(n+1)^2$ vertices, each of which is associated with a Bernoulli random weight w , where $\mathbb{P}(w = 1) = s = 1 - \mathbb{P}(w = 0)$, $0 < s < 1$, and all the weights are independent. The last passage time $T(n, n)$ is the maximum of the sums of all the weights along all unit-step up-right paths on the grid, from $(0, 0)$ to (n, n) . For convenience, the path is considered *left-open-right-closed*, i.e., the weight on $(0, 0)$ is excluded:

$$T(n, n) = \max_{\pi \in \Pi} \sum_{v \in \pi \setminus (0,0)} w(v),$$

where Π is the set of all unit-step up-right paths from $(0, 0)$ to (n, n) , and where each unit-step up-right path $\pi \in \Pi$ is viewed as an ordered set of vertices, i.e., $\pi = \{v_0 = (0, 0), v_1, \dots, v_{2n} = (n, n)\}$ such that $v_{i+1} - v_i$ ($i \in [2n-1]$) is either $\mathbf{e}_1 := (1, 0)$ or $\mathbf{e}_2 := (0, 1)$, and $w : v \rightarrow w(v) \in \{0, 1\}$ is the random weight associated with the vertex $v \in [n] \times [n]$, where $[n] := \{0, 1, 2, \dots, n\}$. Hereafter *directed path* is short for unit-step up-right path and any directed path realizing the last passage time is called a *geodesic*. We also use the notation $T(V_1, V_2)$ to denote the directed last passage time for a rectangular grid from the lower-left vertex V_1 to the upper-right vertex V_2 ($w(V_1)$ is also excluded) and sometimes use coordinates to express V_1 and V_2 , e.g., when $V_1 = (i, j)$ and $V_2 = (k, l)$, $T(V_1, V_2) := T((i, j), (k, l))$.

Let us now briefly describe the content of the paper: in the next section, we present properties of the shape function of DLPP and state our main result (Theorem 1.2.5). Section 1.3 first introduces a way of decomposing the entire grid into blocks in such a way that,

with high probability, most of the blocks in any optimal decomposition are close-to-square shaped. Next, an intermediate rate of convergence result used in the proof of the main theorem is further obtained. Finally, we exhibit two lines ℓ_1 and ℓ_2 respectively above and below the main diagonal, bounding a sector within which, with high probability, geodesics are confined. Then, by finely tuning the slopes of these two bounding lines, we produce a concentration inequality for the fluctuations of the geodesics away from the main diagonal. In the concluding Section 1.4, extensions are briefly stated for the geodesics in directed first passage percolation (DFPP). Then, the case of LCSs is presented and some potential refinements are also discussed.

1.2 Preliminaries and Main Results

In this section, we introduce the shape function g and a modification g_\perp (g -perp) along with some of their properties. It is well known that, by superadditivity and Fekete's Lemma, the non-negative limit

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}T(nx, ny)}{n} = \limsup_{n \rightarrow \infty} \frac{\mathbb{E}T(nx, ny)}{n} := g(x, y)$$

exist for any $x, y \in \mathbb{R}^+$. The function g is typically called the shape function, and by a further application of superadditivity, it can be shown to be concave (see [29]). Instead of studying g directly, we are more interested in its orthogonal modification, i.e., in the function g_\perp , given by $g_\perp(q) = g(1 - q, 1 + q)$, where $q \in (-1, +1)$. Since the transformation $(1 - q, 1 + q)$ is linear, it is trivial to transfer results from g to g_\perp . Therefore, from [29]:

Proposition 1.2.1. *g_\perp is non-negative and concave.*

By the invariance of g under any permutation of its coordinates, i.e., since $g(x, y) = g(y, x)$, g_\perp is symmetric about $q = 0$. Also $g_\perp((-1)^+) = g_\perp(1^-) = g(0, 2) = 2s$. Still, by concavity, g_\perp is non-decreasing on $(-1, 0]$ and non-increasing on $[0, 1)$ and so g_\perp attains its maximum at $q = 0$. The uniqueness of this maximum is not guaranteed but would follow from the strict concavity of the shape function g at $(1, 1)$ which has been conjectured, in particular, for i.i.d. Bernoulli weights. To date, strict concavity has not been proved for any weight distribution. However, in the setting of undirected FPP, a class of weight distributions

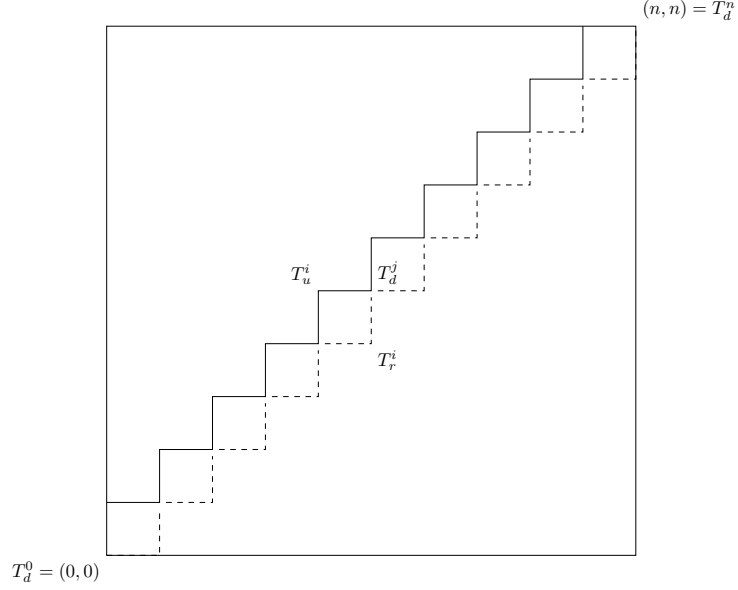


Figure 1: We study the maximum of the passage times along two particular paths, i.e., two paths going along the upper and the lower edges of the 1×1 blocks on the main diagonal.

has been shown (see [14]) to produce a shape function having a flat edge around the direction $(1, 1)$. This class of weights is further studied and more properties of the associated shape function are obtained in [28, 37, 38, 1]. Our first result Theorem 1.3.4 stating that, with probability exponentially close to one, geodesics are bounded away from the upper-left and lower-right corners of the grid, does not require a strict-concavity assumption. Instead, it merely requires the existence of a threshold $t > 0$ such that if $q \in (-1, -t) \cup (t, 1)$, then $g_{\perp}(q) \not\leq g_{\perp}(0) = g(1, 1)$, i.e., that g_{\perp} is not identically constant on $(-1, 1)$. Before tackling this threshold problem, let us better estimate $g_{\perp}(0) = g(1, 1)$.

First, it is clear that any directed path from $(0, 0)$ to (n, n) in a $n \times n$ grid covers exactly $2n$ vertices and the expected passage time associated with up-right path is $2ns$. But, clearly, the passage time associated with any such up-right path is at most the last passage time. Thus $\mathbb{E}T(n, n) \geq 2ns$. Therefore, $g_{\perp}(0) \geq 2s$, however this lower bound is strict.

Lemma 1.2.2. $g_{\perp}(0) - 2s \geq s(1 - s)$.

Proof. Consider the diagonal blocks in the $n \times n$ table, i.e., the n blocks of size 1×1 on the diagonal as in Figure 1. Any up-right path on this block goes either up-right or right-up. Denote by T_u^i the weight associated with the vertex at the upper-left corner of the i th 1×1

diagonal block, while T_r^i is the weight associated with the corresponding lower-right corner for $i \in [n-1]$ and T_d^j is the weight associated with the vertex on the diagonal for $j \in [n]$. Then, all these $3n$ random weights are i.i.d. Bernoulli random variables with parameter s . Moreover, the maximal passage time of all the paths going inside these blocks is a lower bound for the last passage time, i.e.,

$$T(n, n) \geq \sum_{j=1}^n T_d^j + \sum_{i=0}^{n-1} (T_u^i \vee T_r^i).$$

Hence,

$$\begin{aligned} g(1, 1) &\geq \lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E} \left(\sum_{j=1}^n T_d^j + \sum_{i=0}^{n-1} (T_u^i \vee T_r^i) \right) \\ &\geq \lim_{n \rightarrow \infty} \frac{1}{n} (ns + n(1 - (1-s)^2)) \\ &= 3s - s^2. \end{aligned}$$

□

An explicit expression for g is known for geometric or exponential weights but not for Bernoulli weights (e.g., see [36, 21, 35]). So to obtain a specific threshold t , as described above, we combine the lower bound on $g_{\perp}(0)$ obtained in Lemma 1.2.2 with an upper bound on g obtained in [29].

Proposition 1.2.3. *Let $t = 1 - (g(1, 1) - 2s)^2 / 8s(1-s) < 1 - s(1-s)/8$. Then, for any $q \in (-1, -t) \cup (t, 1)$, $g_{\perp}(q) \lesssim g_{\perp}(0) = g(1, 1)$.*

Proof. First by Lemma 4.1 in [29],

$$g(1, y) \leq (1+y)s + 2\sqrt{y(1+y)}\sqrt{s(1-s)}.$$

Without loss of generality, assume $q > 0$, thus

$$\begin{aligned} g_{\perp}(q) &= (1+q)g\left(\frac{1-q}{1+q}, 1\right) \\ &= (1+q)g\left(1, \frac{1-q}{1+q}\right) \\ &\leq (1+q) \left(\frac{2}{1+q}s + 2\sqrt{\frac{2(1-q)}{(1+q)^2}}\sqrt{s(1-s)} \right) \\ &= 2s + 2\sqrt{2-2q}\sqrt{s(1-s)}. \end{aligned} \tag{1.2.1}$$

When $t = 1 - (g(1, 1) - 2s)^2/8(1 - s)s$, the upper bound on $g_\perp(q)$ given in (1.2.1) is equal to $g_\perp(0) = g(1, 1)$. Moreover, by Lemma 1.2.2,

$$g(1, 1) - 2s = g_\perp(0) - 2s \geq s(1 - s),$$

which implies that $t \leq 1 - s(1 - s)/8$. \square

It is commonly believed that, for Bernoulli weights, with sufficiently small parameter, g is strictly concave. In our setting instead, and in order to obtain our main result, the finiteness of the curvature power in the $(1, 1)$ direction is imposed. This assumption is stronger than strict concavity since the curvature power of the shape function is defined as:

Definition 1.2.4. The shape function g is said to have curvature power $\kappa(\mathbf{e})$ at $\mathbf{e} \in \mathbb{R}^+ \times \mathbb{R}^+$, if g is differentiable at \mathbf{e} and there exists $\delta > 0$, such that for any $\mathbf{z} \in \mathbb{R} \times \mathbb{R}$ such that $|\mathbf{z}| < \delta$ and $\mathbf{z} + \mathbf{e}/g(\mathbf{e}) \in L(\mathbf{e})$, where $L(\mathbf{e})$ is a supporting line for g at \mathbf{e} ,

$$c|\mathbf{z}|^{\kappa(\mathbf{e})} \leq |g(\mathbf{e} + \mathbf{z}) - g(\mathbf{e})| \leq C|\mathbf{z}|^{\kappa(\mathbf{e})},$$

for some positive constants c and C depending only on δ . Otherwise, if g is not differentiable at \mathbf{e} , set $\kappa(\mathbf{e}) = 1$. Hereafter, κ is short for $\kappa((1, 1))$.

By symmetry, the supporting line $L(1, 1)$ is in the direction of $(1, -1)$. Hence, the definition of κ is equivalent to the fact that there exists $\delta > 0$, such that for any $\mathbf{z} \in \mathbb{R} \times \mathbb{R}$ satisfying $\mathbf{z} \cdot (1, 1) = 0$ and $|\mathbf{z}| < \delta$,

$$c|\mathbf{z}|^\kappa \leq |g((1, 1) + \mathbf{z}) - g(1, 1)| \leq C|\mathbf{z}|^\kappa,$$

for some positive constants c and C depending only on δ . Then, requiring that $1 \leq \kappa < +\infty$ is in turn equivalent to: there exists $\delta > 0$ such that for any $q \in (-1, 1)$ with $|q| < \delta$,

$$c|q|^\kappa \leq |g_\perp(q) - g_\perp(0)| \leq C|q|^\kappa, \tag{1.2.2}$$

for some positive constants c and C , depending only on δ .

As already indicated, it is believed (e.g., see [24]) that when $s < s_c$, where s_c is the critical probability for the directed last passage percolation with i.i.d. Bernoulli weights, the

shape function g has curvature power $\kappa = 2$. But as mentioned before Lemma 1.2.2, in FPP, a class of weights has been shown to be such that g has flat edges, i.e., there exist infinitely many \mathbf{e} such that $\kappa(\mathbf{e}) = \infty$ (see [14]). It was first proved in [30] that there exists $q \in (-1, 1)$ such that the lower inequality in (1.2.2) holds when $\kappa(1 - q, 1 + q) = 2$, while [11] shows that there is a (possibly different) q at which the upper inequality in (1.2.2) holds when $\kappa(1 - q, 1 + q) = 2$. To finish this section, we state the main result of this chapter.

Theorem 1.2.5. *Let the curvature power κ of the shape function g at $(1, 1)$ be such that $1 \leq \kappa < +\infty$. Then, in a $n \times n$ grid, with probability exponentially close to 1, all the geodesics are within the cylinder, centered on the main diagonal and of width $O(n^{\frac{2\kappa+2}{2\kappa+3}} \sqrt{\ln n})$.*

As far as notations are concerned, and as usual, $a_n = O(b_n)$ is short for there exists a positive constant C such that $|a_n| \leq C|b_n|$, for n large enough; $a_n = \Theta(b_n)$ is short for there exist $0 < c < C < +\infty$ such that $cb_n \leq a_n \leq Cb_n$, for n large enough; $a_n = \Omega(b_n)$ is short for there exists a constant $K > 0$ such that $a_n \geq Kb_n$, for n large enough and finally, $a_n = o(b_n)$ is short for $\lim_{n \rightarrow +\infty} |a_n|/|b_n| = 0$.

1.3 Proof of Main Results

In this section we start by introducing our main tools, i.e., decompositions and blocks, and then prove some concentration results which are further related to the concentration of geodesics needed to obtain our main result.

1.3.1 Blocks, Decompositions and Concentration

Throughout the rest of this manuscript, let $n = mk$ so that the x -axis of the grid is divided into m segments each of equal length k . Meanwhile, the y -axis of the grid is also divided into m segments. The $(m + 1)$ -tuples $\vec{r} = (r_0, r_1, \dots, r_m)$ made of the end points of these consecutive segments on the y -edges, i.e.,

$$r_0 = 0 \leq r_1 \leq r_2 \leq \dots \leq r_{m-1} \leq r_m = n,$$

is called a *decomposition* of the y -axis. This decomposition leads to a decomposition of the grid into m rectangular blocks, of which the i th ($i = 1, 2, \dots, m$) block has lower-left corner

$((i-1)k, r_{i-1})$ and upper-right corner (ik, r_i) , and is of size $k \times (r_i - r_{i-1})$. Moreover, the last passage time associated with the decomposition \vec{r} is defined as the summation over all the m last passage times in these m blocks, i.e.,

$$T_n(\vec{r}) = \sum_{i=1}^m T(((i-1)k, r_{i-1}), (ik, r_i)).$$

By superadditivity, it is clear that $T(n, n) \geq T_n(\vec{r})$. Moreover, as explained next, there always exists a decomposition \vec{r}_* such that $T(n, p_0 n) = T_n(\vec{r}_*)$, and such a decomposition is called *optimal*. Indeed, one can construct an *optimal* decomposition \vec{r}_* by taking vertices (ik, r_i) , $i = 0, 1, \dots, m$, on a geodesic for the entire $n \times n$ grid. Heuristically, any optimal decomposition \vec{r}_* should, roughly, be evenly distributed over n , i.e., all the m blocks in any optimal decomposition should be mostly square shaped at least with high probability. To be more precise, let us fix $0 < \eta < 1$ and $p_i > 0$ ($i = 1, 2$) such that $0 < p_1 < 1 < p_2$. Let R_{η, p_1, p_2} be the deterministic set of decompositions \vec{r} such that

$$\#\{i \in [m] : kp_1 \leq r_i - r_{i-1} \leq kp_2\} \geq (1 - \eta)m, \quad (1.3.1)$$

in words R_{η, p_1, p_2} represents the decompositions having a proportion of at least $(1 - \eta)$ of those m blocks close-to-square shaped, i.e., the decompositions for which the slope of the block diagonal is close to 1, i.e., the non-skewed decompositions. Finally, let A_{η, p_1, p_2}^n be the event that all the optimal decompositions are in R_{η, p_1, p_2} , i.e., if \vec{r} is optimal, then

$$\vec{r} \in R_{\eta, p_1, p_2}. \quad (1.3.2)$$

Next, we show a lemma asserting that for any decomposition $\vec{r} = (r_0, r_1, \dots, r_m) \in R_{\eta, p_1, p_2}^c$, the difference between the expected overall last passage time and the expected last passage times associated with \vec{r} is at least linear in n . Before proving it, it is shown that even when the vertex weights, belonging to a particular set to be specified, are independently resampled, the absolute change in the last passage time can be upper-bounded by 1. To specify such a set, declare the vertices $\{V_i = (X_i, Y_i)\}_{i=1}^k$ to be strictly decreasing, if there exists a permutation π of $\{1, 2, \dots, k\}$ such that

$$V_{\pi(1)} \prec V_{\pi(2)} \prec \dots \prec V_{\pi(k)},$$

where $V_i \prec V_j$ indicates that both $X_i < X_j$ and $Y_i < Y_j$. For example, on a $n \times n$ grid, the set of all the vertices on the reversed diagonal, i.e., $\{(n-i, i)\}_{i=0}^n$ is a strictly decreasing set and its cardinality is $n+1$.

Lemma 1.3.1. *Let a rectangular grid have lower-left vertex V_1 and upper-right vertex V_2 and let S be a strictly decreasing set of vertices on the grid. Then, the absolute difference between the last passage times in the original weights setting and in the modified weights setting, where the weights on S are independently resampled, is upper bounded by 1, i.e.,*

$$|T(V_1, V_2) - T^S(V_1, V_2)| \leq 1,$$

where $T(V_1, V_2)$ and $T^S(V_1, V_2)$ are respectively the last passage times before and after resampling.

Proof. Let Π be the set of all up-right paths from V_1 to V_2 . Since S is a set of strictly decreasing vertices, for any path $\pi \in \Pi$ viewed as a set of vertices, the intersection between π and S is either empty or contains exactly one element, i.e., $\#(\pi \cap S) \leq 1$. Thus,

$$T_\pi^S(V_1, V_2) - T_\pi(V_1, V_2) \leq 1,$$

where the upper bound is 1 if and only if there is a vertex $v \in \pi \cap S$ such that $w(v) = 0$ and $w^S(v) = 1$. Let π_*^S be a geodesic after resampling S , i.e., $T_{\pi_*^S}^S(V_1, V_2) = T^S(V_1, V_2)$. It follows that

$$\begin{aligned} T^S(V_1, V_2) &= T_{\pi_*^S}^S(V_1, V_2) \\ &\leq T_{\pi_*^S}(V_1, V_2) + 1 \\ &\leq \max_{\pi \in \Pi} T_\pi(V_1, V_2) + 1 \\ &\leq T(V_1, V_2) + 1. \end{aligned}$$

Symmetrically, $T(V_1, V_2) - T^S(V_1, V_2) \leq 1$ and thus

$$|T(V_1, V_2) - T^S(V_1, V_2)| \leq 1.$$

□

For further convenience, we introduce a transformed shape function g_{\wedge} which depends on the slope of the main diagonal of the grid. Specifically, for $p > 0$, set

$$g_{\wedge}(p) := \lim_{n \rightarrow \infty} \frac{\mathbb{E}T(n, np)}{n(1+p)/2}.$$

Now, recalling that $g_{\perp} : q \in (-1, 1) \rightarrow g_{\perp}(q) \in (0, \infty)$ is defined via

$$g_{\perp}(q) = g(1-q, 1+q) = \lim_{n \rightarrow \infty} \frac{\mathbb{E}T(n-nq, n+nq)}{n},$$

it is clear that

$$g_{\wedge}(p) = g_{\perp}\left(\frac{p-1}{p+1}\right),$$

for $p \in (0, +\infty)$. To prove a result showing that the difference of the expectations is at least linear in n , in addition to Lemma 1.3.1, a rate of convergence result for $\mathbb{E}T_n/n$ is also needed. This is stated and proved next, with a proof adapted from [33].

Proposition 1.3.2. $0 \leq g_{\wedge}(1) - \mathbb{E}T_n/n \leq c\sqrt{\ln n/n}$, where $c > 0$ is an absolute constant.

Proof. Consider the last passage time T_{kn} of site percolation on a $kn \times kn$ grid. A sequence of vertices $\vec{V} = (V_1 = (X_1 = 0, Y_1 = 0), V_2, \dots, V_k = (X_k = kn, Y_k = kn))$ is called a *partition* of the grid, if

$$0 = X_1 \leq X_2 \leq \dots \leq X_k = kn, \quad (1.3.3)$$

$$0 = Y_1 \leq Y_2 \leq \dots \leq Y_k = kn, \quad (1.3.4)$$

$$\|V_i - V_{i+1}\|_1 = 2n, \quad (1.3.5)$$

where $\|\cdot\|_1$ denotes the ℓ_1 -distance. Further, let the last passage time associated with some partition \vec{V} be

$$T(\vec{V}) = \sum_{i=0}^{k-1} T(V_i, V_{i+1}).$$

Then, as proved next,

$$T_{kn} = \max_{\text{partitions } \vec{V}} T(\vec{V}). \quad (1.3.6)$$

First, it is clear that the identity is true if only (1.3.3) and (1.3.4) are imposed on partitions. To show it is fine to include (1.3.5), it suffices to show that any geodesic can be divided into k segments such that the ℓ_1 -distance between two ends of any segment is exactly $2n$. Assume

some geodesic is an ordered set of $2kn+1$ vertices ($W_0 = (0, 0), W_2, W_3, \dots, W_{2kn} = (kn, kn)$). Notice that $\|W_i - W_{i+j}\|_1 = j$, for any $i, i+j \in [2kn]$. Therefore, this geodesic can be divided on ($V_0 = W_0, V_1 = W_{2n}, \dots, V_k = W_{2kn}$) into k segments with $\|V_i - V_{i+1}\|_1 = 2n$.

Next, consider a particular set of directed paths going from $(0, 0)$, through $(k, 2n-k)$, to $(2n, 2n)$ on a $2n \times 2n$ grid. Then, by superadditivity, $T(k, 2n-k) + T(2n-k, k) \leq T(2n, 2n)$. Further, thanks to symmetry, $\mathbb{E}T(k, 2n-k) = \mathbb{E}T(2n-k, k)$. Hence, $\mathbb{E}T(k, 2n-k) \leq \frac{1}{2}\mathbb{E}T(2n, 2n)$. So, $\mathbb{E}T(\vec{V}) \leq k\mathbb{E}T(2n, 2n)/2$.

On the other hand, let us view $T(\vec{V})$ as a function

$$T(\vec{V}) : (D_1, \dots, D_{2kn}) \rightarrow T(\vec{V})(D_1, \dots, D_{2kn}) \in \mathbb{N},$$

where $\{D_j\}_{j=1}^{2kn}$ is the set of batches of the weights $w(v)$ on the same reversed diagonal, i.e., $D_j = \{w(v) \mid v \in \{x+y=j\} \cap [kn] \times [kn]\}$. Clearly, the independence of the weights yields the independence of the random vectors $D_j, j = 1, \dots, 2kn$. Further, any batch D_j is a strictly decreasing set of vertices and so by Lemma 1.3.1, independently resampling any one of these random vectors, say, as D'_{j_0} gives

$$|T(\vec{V})(D_1, \dots, D'_{j_0}, \dots, D_{2kn}) - T(\vec{V})(D_1, \dots, D_{j_0}, \dots, D_{2kn})| \leq 1.$$

Further, applying Hoeffding's martingale inequality gives

$$\mathbb{P}\left(T(\vec{V}) - \mathbb{E}T(\vec{V}) \geq tkn\right) \leq \exp(-t^2kn),$$

and so,

$$\mathbb{P}\left(T(\vec{V}) - \frac{k}{2}\mathbb{E}T_{2n} \geq tkn\right) \leq \exp(-t^2kn).$$

In addition, by (1.3.6),

$$\begin{aligned} \mathbb{P}\left(\frac{T_{kn}}{kn} - \frac{\mathbb{E}T_{2n}}{2n} \geq t\right) &= \mathbb{P}\left(T_{kn} - \frac{k}{2}\mathbb{E}T_{2n} \geq tkn\right) \\ &\leq \sum_{\text{partitions } \vec{V}} \mathbb{P}\left(T(\vec{V}) - \frac{k}{2}\mathbb{E}T_{2n} \geq tkn\right) \\ &\leq \#\text{partitions} \exp(-t^2kn). \end{aligned} \tag{1.3.7}$$

Since $\#partitions = \binom{kn+k-1}{k-1}$ and $\binom{p}{q} \leq p^p/q^q(p-q)^{p-q}$,

$$\#partitions \leq \binom{kn+k}{k} \leq (kn+k)^{kn+k}/k^k(kn)^{kn} \leq \exp(ck \ln n),$$

for some absolute constant $c > 0$. Combining this with (1.3.7) and taking $t = \sqrt{2c \ln n/n}$ leads to

$$\mathbb{P} \left(\frac{T_{kn}}{kn} - \frac{\mathbb{E}T_{2n}}{2n} \geq \sqrt{\frac{2c \ln n}{n}} \right) \leq \exp(-ck \ln n).$$

Hence,

$$\frac{\mathbb{E}T_{kn}}{kn} - \frac{\mathbb{E}T_{2n}}{2n} \leq \sqrt{\frac{2c \ln n}{n}} + 2 \exp(-ck \ln n),$$

and letting $k \rightarrow \infty$ gives,

$$\frac{\mathbb{E}T_{2n}}{2n} \geq \gamma^* - \sqrt{\frac{2c \ln n}{n}}.$$

In addition, for odd integers,

$$\frac{\mathbb{E}T_{2n+1}}{2n+1} \geq \frac{\mathbb{E}T_{2n}}{2n} - \frac{\mathbb{E}T_{2n+1}}{2n(2n+1)} \geq \gamma^* - \sqrt{\frac{2c \ln n}{n}} - \frac{1}{n}.$$

□

To state our next lemma, recall that $R_{\eta, p_1, p_2} = \{r : \#\{i \in [m] : kp_1 \leq r_i - r_{i-1} \leq kp_2\} \geq (1-\eta)m\}$.

Lemma 1.3.3. *Let $0 < \eta < 1$ and let p_i ($i = 1, 2$) be such that $0 < p_1 < 1 < p_2$, $g_{\wedge}(p_i) < g_{\wedge}(1)$. Let $\delta^* = \min(g_{\wedge}(1) - g_{\wedge}(p_1), g_{\wedge}(1) - g_{\wedge}(p_2))$ and let $\delta^*\eta = \Omega(\sqrt{\log n/n})$. Then, for any $\vec{r} = (r_0, r_1, \dots, r_m) \in R_{\eta, p_1, p_2}^c$ and any $\delta \in (0, \delta^*)$,*

$$\mathbb{E}(T_n(\vec{r}) - T_n) \leq -\frac{\delta\eta n}{2}, \tag{1.3.8}$$

for all $n = n(\eta, \delta)$ large enough.

Proof. Let $p > 0$. By superadditivity, $g_{\wedge}(p)$ is well defined and finite. Moreover, for any $k \geq 1$,

$$\frac{2\mathbb{E}T(k, kp)}{k(1+p)} \leq g_{\wedge}(p). \tag{1.3.9}$$

Since g_{\perp} is symmetric around $q = 0$ and concave and since $(1 - p)/(1 + p)$ is a monotone transformation in p , g_{\wedge} is non-decreasing up to $p = 1$ and non-increasing thereafter. Proposition 1.2.3 shows that there exist $0 < p_1 < 1 < p_2$ such that for any $p \notin [p_1, p_2]$,

$$g_{\wedge}(p) \leq \max(g_{\wedge}(p_1), g_{\wedge}(p_2)). \quad (1.3.10)$$

Therefore, for any $p \notin [p_1, p_2]$, (1.3.9) and (1.3.10) lead to:

$$\frac{2\mathbb{E}T(k, kp)}{k(1+p)} \leq \max(g_{\wedge}(p_1), g_{\wedge}(p_2)) = g_{\wedge}(1) - \delta^*, \quad (1.3.11)$$

where $0 < \delta^* := \min(g_{\wedge}(1) - g_{\wedge}(p_1), g_{\wedge}(1) - g_{\wedge}(p_2))$.

From here on, the proof proceeds as the proof, with its notation, of Lemma 2.1, in [19]. Since the weights are identically distributed, in the i th block $[(i-1)k+1, ik] \times [r_{i-1}+1, r_i]$, letting $r_i - r_{i-1} := kp$ and assuming $(r_i - r_{i-1})/k = p \notin [p_1, p_2]$, then (1.3.11) gives

$$g_{\wedge}(1) - \frac{2\mathbb{E}T(((i-1)k+1, r_{i-1}+1), (ik, r_i))}{k + r_i - r_{i-1}} \geq \delta^*. \quad (1.3.12)$$

Hence,

$$\frac{1}{2}g_{\wedge}(1)(k + r_i - r_{i-1}) - \mathbb{E}T(((i-1)k+1, r_{i-1}+1), (ik, r_i)) \geq \frac{1}{2}\delta^*k.$$

Letting $\mathcal{M} := \{i : r_i - r_{i-1} \notin [kp_1, kp_2]\}$, we then have

$$\sum_{i \in \mathcal{M}} \frac{1}{2}g_{\wedge}(1)(k + r_i - r_{i-1}) - \mathbb{E}T(((i-1)k+1, r_{i-1}+1), (ik, r_i)) \geq \frac{1}{2}\delta^*k\eta m = \frac{1}{2}n\delta^*\eta, \quad (1.3.13)$$

while, for any $i \in \mathcal{M}^c = \{i : r_i - r_{i-1} \in [kp_1, kp_2]\}$, (1.3.9) gives

$$\frac{1}{2}g_{\wedge}(1)(k + r_i - r_{i-1}) - \mathbb{E}T(((i-1)k+1, r_{i-1}+1), (ik, r_i)) \geq 0.$$

Therefore,

$$\sum_{i \in \mathcal{M}} \frac{1}{2}g_{\wedge}(1)(k + r_i - r_{i-1}) - \mathbb{E}T(((i-1)k+1, r_{i-1}+1), (ik, r_i)) \leq g_{\wedge}(1)n - \mathbb{E}T_n(\vec{r}). \quad (1.3.14)$$

Combining (1.3.13) and (1.3.14) leads to,

$$g_{\wedge}(1)n - \mathbb{E}T_n(\vec{r}) \geq \frac{n\delta^*\eta}{2}, \quad (1.3.15)$$

when $\vec{r} = (r_0, r_1, \dots, r_m) \in R_{\eta, p_1, p_2}^c$. Next, by Proposition 1.3.2 and since $\delta^*\eta = \Omega(\sqrt{\ln n/n})$,

it follows that for any positive $\delta < \delta^*$ and for all n large enough,

$$0 \leq g_{\wedge}(1) - \frac{\mathbb{E}T_n}{n} \leq c\sqrt{\frac{\ln n}{n}} \leq \frac{(\delta^* - \delta)\eta}{2}, \quad (1.3.16)$$

where $c > 0$ is an absolute constant. So combining (1.3.15) and (1.3.16), and for $\vec{r} = (r_0, r_1, \dots, r_m) \in R_{\eta, p_1, p_2}^c$,

$$\mathbb{E}(T_n(\vec{r}) - T_n) \leq -\frac{\delta \eta n}{2}.$$

□

Before presenting the main result of this section, recall (see (1.3.2)) that A_{η, p_1, p_2}^n is the event that all the optimal decompositions belong to R_{η, p_1, p_2} .

Theorem 1.3.4. *Let $0 < \eta < 1$ and let p_i ($i = 1, 2$) be such that $0 < p_1 < 1 < p_2$, $g_{\wedge}(p_i) < g_{\wedge}(1)$. Let $\delta^* = \min(g_{\wedge}(1) - g_{\wedge}(p_1), g_{\wedge}(1) - g_{\wedge}(p_2))$ and let $\delta^* \eta = \Omega(\sqrt{\log n/n})$. Let the integer k be such that $(1 + \ln k)/k \leq \delta^2 \eta^2 / 16$, where $\delta \in (0, \delta^*)$. Then,*

$$\mathbb{P}(A_{\eta, p_1, p_2}^n) \geq 1 - \exp\left(-n \left(-\frac{1 + \ln k}{k} + \frac{\delta^2 \eta^2}{16}\right)\right),$$

for all $n = n(\eta, \delta)$ large enough.

Proof. The beginning of this proof, which is similar to that of Theorem 2.2 in [19], is only sketched. By superadditivity, the decomposition \vec{r} is optimal if and only if

$$T_n(\vec{r}) \geq T_n. \quad (1.3.17)$$

Assume now that the event A_{η, p_1, p_2}^n does not hold. Then there exists an optimal decomposition \vec{r}_* such that $\vec{r}_* \in R_{\eta, p_1, p_2}^c$, i.e.,

$$\begin{aligned} (A_{\eta, p_1, p_2}^n)^c &= \bigcup_{\vec{r} \in R_{\eta, p_1, p_2}^c} \{\vec{r} = \vec{r}_*\} \\ &= \bigcup_{\vec{r} \in R_{\eta, p_1, p_2}^c} \{T_n(\vec{r}) - T_n \geq 0\}, \end{aligned}$$

hence,

$$\mathbb{P}((A_{\eta, p_1, p_2}^n)^c) \leq \sum_{\vec{r} \in R_{\eta, p_1, p_2}^c} \mathbb{P}(T_n(\vec{r}) - T_n \geq 0). \quad (1.3.18)$$

Then, by Lemma 1.3.3, for any decomposition $\vec{r} \in R_{\eta, p_1, p_2}^c$,

$$\mathbb{E}(T_n(\vec{r}) - T_n) \leq -\frac{\delta \eta n}{2},$$

and so

$$\mathbb{P}(T_n(\vec{r}) - T_n \geq 0) \leq \mathbb{P}\left(T_n(\vec{r}) - T_n - \mathbb{E}(T_n(\vec{r}) - T_n) \geq \frac{\delta\eta n}{2}\right), \quad (1.3.19)$$

for all n large enough. Next, as in the proof of Proposition 1.3.2, we view the random variable $T_n(\vec{r}) - T_n := \Delta$ as a function

$$\Delta : (D_1, \dots, D_{2n}) \rightarrow \Delta(D_1, \dots, D_{2n}) \in \mathbb{Z} \cap [-2n, 2n],$$

where $\{D_j\}_{j=1}^{2n}$ is the set of batches of the weights $w(v)$ on the same reversed diagonal, i.e., $D_j = \{w(v) \mid v \in \{x + y = j\} \cap [n] \times [n]\}$. So by Lemma 1.3.1 again, independently resampling any one of these random vectors, say, as D'_{j_0} gives

$$\begin{aligned} |\Delta(D_1, \dots, D'_{j_0}, \dots, D_{2n}) - \Delta(D_1, \dots, D_{j_0}, \dots, D_{2n})| \\ \leq |T_n^{j_0}(\vec{r}) - T_n(\vec{r})| + |T_n^{j_0} - T_n| \leq 2, \end{aligned} \quad (1.3.20)$$

where $T_n^{j_0}(\vec{r})$ and $T_n^{j_0}$ are respectively the last passage time associated with \vec{r} and the overall last passage time with the weights in D_{j_0} resampled.

Finally, Hoeffding's martingale inequality applied to $\Delta(D_1, \dots, D_{2n})$ and (1.3.19) yield

$$\mathbb{P}(T_n(\vec{r}) - T_n \geq 0) \leq \exp\left(-\frac{\delta^2\eta^2 n}{16}\right),$$

for $\vec{r} \in R_{\eta, p_1, p_2}^c$. Further, by (1.3.18),

$$\begin{aligned} \mathbb{P}((A_{\eta, p_1, p_2}^n)^c) &\leq (ek)^m \exp\left(-\frac{\delta^2\eta^2 n}{16}\right) \\ &\leq \exp\left(-n\left(-\frac{1 + \ln k}{k} + \frac{\delta^2\eta^2}{16}\right)\right), \end{aligned}$$

since

$$\#R_{\eta, p_1, p_2}^c \leq \binom{n}{m} \leq \frac{n^m}{m!} \leq \left(\frac{en}{m}\right)^m,$$

when n is large enough. □

Remark 1.3.5. *Note that above, when applying Hoeffding's martingale inequality and if only a single weight had been independently resampled then, the exponential concentration would have failed to hold, since this naively constructed martingale would have had a length of size $\Theta(n^2)$. This justifies and motivates resampling weights in batches.*

1.3.2 Proof of the Main Result

Heuristically, if most blocks in an optimal decomposition are close-to-square shaped, then all the vertices on the diagonals of these blocks are close to the main diagonal of the grid and therefore all the corresponding geodesics going through these vertices do not deviate much from it. Further, the parameters such as k , δ and η can be fixed in an optimal way so that the cylinder, in which geodesics are confined, is as small as possible.

Proof of Theorem 1.2.5: Let D_{η,p_1,p_2}^n be the event that all the geodesics are above the line $\ell_1: y = p_1x - p_1n\eta - p_1k$ and below the line $\ell_2: y = p_2x + p_2n\eta + p_2k$. We first show that the probability of this event is exponentially close to 1. Again the proof is similar to the corresponding result in [19] and as such only sketched. We start with a few definitions: denote by D_a^n the event that all the geodesics are above the line $\ell_1: y = p_1x - p_1n\eta - p_1k$ and by D_b^n the event that they are below the line $\ell_2: y = p_2x + p_2n\eta + p_2k$. Then $D_{\eta,p_1,p_2}^n = D_a^n \cap D_b^n$, hence

$$\mathbb{P}((D_{\eta,p_1,p_2}^n)^c) \leq \mathbb{P}((D_a^n)^c) + \mathbb{P}((D_b^n)^c).$$

Moreover, as shown next,

$$A_{\eta,p_1,p_2}^n \subset D_a^n, \quad A_{\eta,p_1,p_2}^n \subset D_b^n, \quad (1.3.21)$$

so that by Theorem 1.3.4

$$\mathbb{P}((D_{\eta,p_1,p_2}^n)^c) \leq 2 \exp \left(-n \left(-\frac{1 + \ln k}{k} + \frac{\delta^2 \eta^2}{16} \right) \right). \quad (1.3.22)$$

To prove (1.3.21), at first we prove that $A_{\eta,p_1,p_2}^n \subset D_a^n$. This last inclusion is obtained by considering three cases which depend on x : If (x, y) is on one of the geodesics in the event A_{η,p_1,p_2}^n , namely, $x = uk$, where $u \in \mathbb{N} = \{0, 1, 2, \dots\}$, and $uk \leq n\eta$; $x = uk$, where $u \in \mathbb{N}$ and $uk > n\eta$; and there exists $u \in \mathbb{N}$ such that $uk < x < (u+1)k$. Before we move on to verify the inclusion case by case, recall again that A_{η,p_1,p_2}^n corresponds to geodesic decompositions belonging R_{η,p_1,p_2} , i.e., such that the number of $i \in [m]$ with $(r_{i+1} - r_i) \in [p_1k, p_2k]$ is at least $(1 - \eta)m$, where m is the total number of blocks and $mk = n$ (see (1.3.1)).

In the first of these cases, $p_1x - p_1n\eta \leq 0$ and therefore,

$$y \geq 0 \geq p_1x - p_1n\eta \geq p_1x - p_1n\eta - p_1k.$$

In the second case, by the very definition of R_{η,p_1,p_2} , there are at most ηm blocks having side length $(r_{i+1} - r_i)$ less than $p_1 k$. Since $x = uk$, in the worst case, all these ηm blocks appear among the first u blocks. Hence, at least $u - \eta m$ blocks of the first u blocks have side length at least equal to $p_1 k$. Therefore,

$$y \geq (u - \eta m)p_1 k = p_1(uk) - p_1 \eta m k = p_1 x - p_1 \eta n \geq p_1 x - p_1 \eta n - p_1 k.$$

In the third and last case, since $x_1 := uk < x < (u + 1)k$, then

$$x - x_1 < k. \tag{1.3.23}$$

From the first two cases, $y_1 \geq p_1 x_1 - p_1 \eta n$. Moreover, a geodesic is a directed path and so $y \geq y_1$ since $x > x_1$. Hence, by (1.3.23)

$$y \geq y_1 \geq p_1 x_1 - p_1 \eta n \geq p_1 x - p_1 k - p_1 \eta n.$$

Symmetrically, a reversed inequality can be proved for the upper bounding line $y = p_2 x + p_2 n \eta + p_2 k$, and then (1.3.22) follows.

Now, let $k = n^\alpha$, $p_{1,2} = 1 \pm n^{-\beta}$, for $0 < \alpha, \beta < 1$ and so, $\delta^* = \min(g_\wedge(1) - g_\wedge(p_1), g_\wedge(1) - g_\wedge(p_2)) = cn^{-\kappa\beta}$, for some constant $c > 0$. Further, set $\delta = \delta^*/2 = cn^{-\kappa\beta}/2$ and let $\eta = 4\sqrt{2}n^{\kappa\beta-\alpha/2}\sqrt{1+\alpha\ln n}/c$ in (1.3.22) be such that $2\kappa\beta < \alpha$. Then, the condition $\delta^*\eta = \Theta(n^{-\alpha/2}\sqrt{\ln n}) = \Omega(\sqrt{\ln n/n})$ is satisfied, since $\alpha < 1$. Hence,

$$\mathbb{P}(D_{\alpha,\beta}^n) \geq 1 - 2\exp(-(1 + \alpha \ln n)n^{1-\alpha}),$$

where $D_{\alpha,\beta}^n$ is the event that all the geodesics are above the line $y = (1 - n^{-\beta})(x - cn^{1+\kappa\beta-\alpha/2}\sqrt{1+\alpha\ln n} - n^\alpha)$ and below the line $y = (1 + n^{-\beta})(x + cn^{1+\kappa\beta-\alpha/2}\sqrt{1+\alpha\ln n} + n^\alpha)$.

Lastly, we will fix the orders of α and β so that the cylinder has minimal width and so that the condition $2\kappa\beta < \alpha$ is satisfied. Notice that the distances at which the lines $\ell_{1,2}$ are from the main diagonal is of the same order as the Euclidean distance from their intercepts on the left and right edges of the grid to, respectively, the lower-left vertex V_1 and the upper-right vertex V_2 as pictured in Figure 2. For the lower bounding line ℓ_1 , denoting

its intercept on the left edge by U_1^1 ,

$$\begin{aligned} |U_1^1 V_1| &= (1 - n^{-\beta})(cn^{1+\kappa\beta-\alpha/2}\sqrt{1 + \alpha \ln n} + n^\alpha) \\ &= \Theta(n^{1+\kappa\beta-\alpha/2}\sqrt{\ln n} + n^\alpha). \end{aligned}$$

Then denoting its intercept on the right edge by U_2^1 , whose y -coordinate is $(1 - n^{-\beta})(n - cn^{1+\kappa\beta-\alpha/2}\sqrt{1 + \alpha \ln n} - n^\alpha)$,

$$\begin{aligned} |U_2^1 V_2| &= (1 - n^{-\beta'})n - (1 - n^{-\beta})(n - cn^{1+\kappa\beta-\alpha/2}\sqrt{1 + \alpha \ln n} - n^\alpha) \\ &= n(n^{-\beta} - n^{-\beta'}) + (1 - n^{-\beta})(cn^{1+\kappa\beta-\alpha/2}\sqrt{1 + \alpha \ln n} + n^\alpha) \\ &= \Theta(n^{1-\beta} + n^{1+\kappa\beta-\alpha/2}\sqrt{\ln n} + n^\alpha), \end{aligned}$$

since $\beta' > \beta$. Therefore the distance from ℓ_1 to the diagonal is of order

$$n^{(1-\beta) \vee (1+\kappa\beta-\alpha/2) \vee \alpha} \sqrt{\ln n}.$$

Symmetrically, a similar result holds true for the upper line ℓ_2 . The minimizing order occurs for $1 - \beta = 1 + \kappa\beta - \alpha/2 = \alpha$, i.e.,

$$\alpha = (2\kappa + 2)\beta = \frac{2\kappa + 2}{2\kappa + 3} > 2\kappa\beta.$$

Setting $\alpha = (2\kappa + 2)/(2\kappa + 3)$ and $\beta = 1/(2\kappa + 3)$ in the event $D_{\alpha,\beta}^n$ gives

$$\mathbb{P} \left(D_{\frac{2\kappa+2}{2\kappa+3}, \frac{1}{2\kappa+3}}^n \right) \geq 1 - 2 \exp \left(- \left(1 + \frac{2\kappa+2}{2\kappa+3} \ln n \right) n^{1/(2\kappa+3)} \right),$$

which completes the proof. \square

1.4 Concluding Remarks

By symmetry, it is clear that our methodology for proving the concentration of the geodesics in DLPP is also applicable to the concentration of geodesics in Bernoulli directed first passage site percolation. In DFPP, one studies the minimum of the passage times instead of maximum. In that context, the shape functions g , g_\perp and g_\searrow are convex instead of concave. Then, a version of Lemma 1.3.3 with the inequality (1.3.8) reversed holds true. Further, a version of Lemma 1.3.1 replacing last passage time by first passage time is still true. Then,

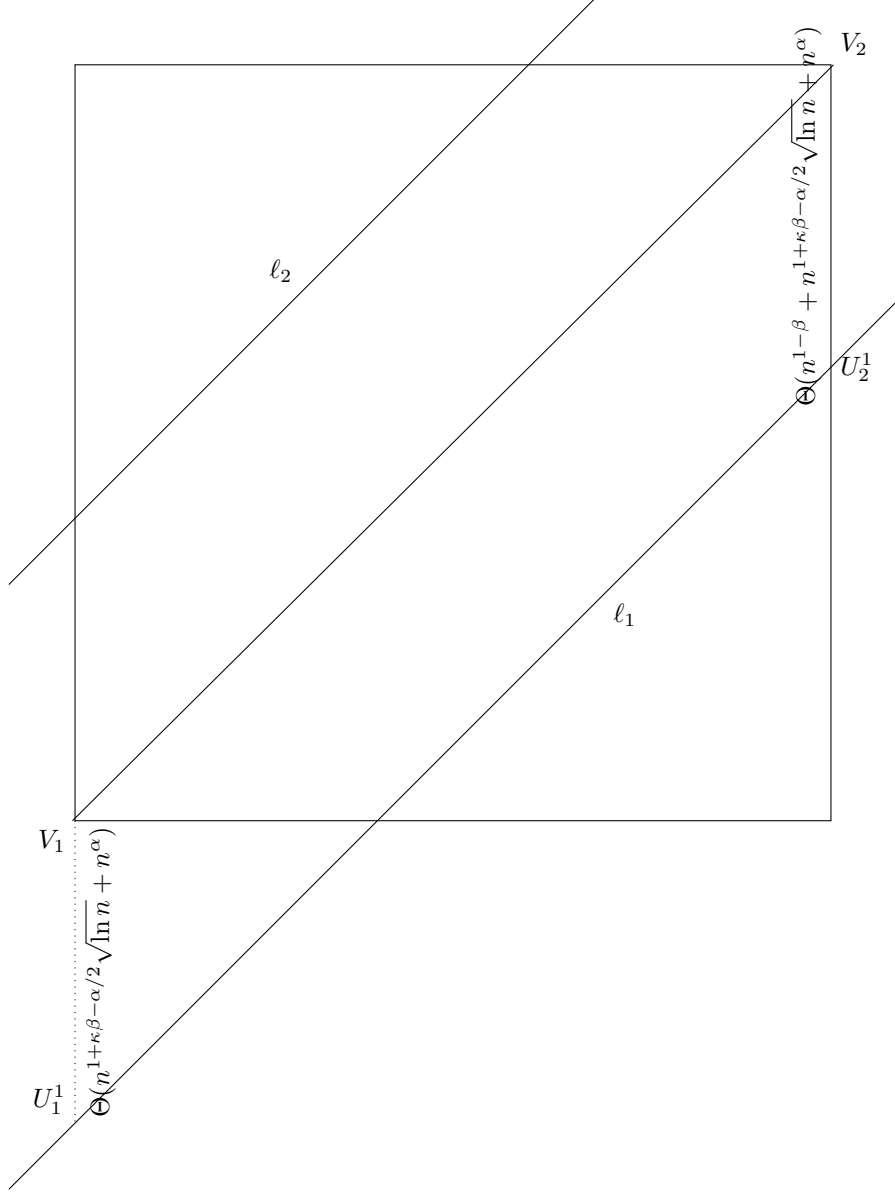


Figure 2: With high probability, the geodesics deviate from the main diagonal by an amount at most of order $\Theta(n^{1-\beta} + n^{1+\kappa\beta-\alpha/2}\sqrt{\ln n} + n^\alpha)$

so are Theorem 1.3.4, Proposition 1.2.3, Proposition 1.3.2 and Theorem 1.2.5. Combining all these results finally leads to:

Theorem 1.4.1. *In directed Bernoulli first passage site percolation, let the curvature power κ of the shape function g at $(1,1)$ be such that $1 \leq \kappa < +\infty$. Then, in a $n \times n$ grid, with probability exponentially close to 1, all the geodesics are within the cylinder, centered on the main diagonal and of width $O(n^{\frac{2\kappa+2}{2\kappa+3}}\sqrt{\ln n})$.*

To gain a better intuitive view of the concentration order, let, as commonly believed, $\kappa = 2$. Then, the order is $O(n^{6/7}\sqrt{\ln n})$. Again, it is conjectured that the correct order should be $O(n^{2/3})$ and a currently available bound for the exponent ξ is $3/4$, which has been shown in [30], in the setting of first passage percolation on grids in arbitrary dimension.

It is further worth mentioning that our methodology can also be adapted to produce the order of the closeness to the diagonal for the optimal alignments corresponding to the LCSs of two random words of size n . In that setting, it is known that the curvature power of the shape function of the LCSs at $(1,0)$ and $(0,1)$ is equal to 1 (see the proof of Lemma 2.1 in [19]). However, the value of κ (the curvature power at $(1,1)$) remains unknown but we conjecture it to be equal to 2, as in the percolation models. Adapting our methods leads to:

Theorem 1.4.2. *In the longest common subsequences problem, let the curvature power κ of the shape function g at $(1,1)$ be such that $1 \leq \kappa < +\infty$. Then, with probability exponentially close to 1, all the alignments corresponding to the longest common subsequences of two random words of length n are within the cylinder, centered on the main diagonal and of width of order $O(n^{\frac{2\kappa+2}{2\kappa+3}}\sqrt{\ln n})$.*

Let the exponent of transversal fluctuations ξ be:

$$\xi = \inf\{\gamma > 0 : \liminf_{n \rightarrow +\infty} \mathbb{P}(A_n^\gamma) = 1\},$$

where A_n^γ is the event that all the optimal alignments are confined to a cylinder centered on the main diagonal and of width of order n^γ . Therefore, from Theorem 1.4.2, for LCSs, $\xi \leq (2\kappa + 2)/(2\kappa + 3)$. Moreover, as previously mentioned, the shape fluctuations exponent for LCSs has been shown to be $\chi = 1/2$, i.e., $\text{Var}(LC_n) = \Theta(n)$, for various asymmetric

discrete distribution on any finite dictionary (see [18, 15, 27, ...]). But, by the conjectured KPZ universality relation with curvature power κ ,

$$\chi = \kappa\xi - (\kappa - 1).$$

This leads, for $\chi = 1/2$, to $\xi = (2\kappa - 1)/(2\kappa)$ which we conjecture to be equal to $3/4$.

Chapter II

POWER LOWER BOUNDS FOR THE CENTRAL MOMENTS OF THE LAST PASSAGE TIME FOR DIRECTED PERCOLATION IN A THIN RECTANGLE

2.1 *Introduction and Statements of Results*

Longitudinal/shape fluctuations, i.e., the standard deviation of first/last passage time, has attracted a lot of attention in the study of percolation systems. It is conjectured that, on a two dimensional $n \times n$ -grid, the fluctuations are of order $n^{1/3}$ in undirected/directed first/last passage percolation, with various weight distributions satisfying moment conditions. However, this result has presently only been proved under exponential or geometric weights, e.g., see [22, 23, 4], and to date, for general weight distributions satisfying moments conditions, only an upper bound of sublinear order $O(\sqrt{n/\ln n})$ (see [24, 25, 6, 7, 12]) and a lower bound of order $o(\sqrt{\ln n})$ (see [31, 30, 37, 1, ?]) have been obtained for first passage percolation in various dimensions. More is known for the directed last passage time (*DLPP*) in a thin rectangular lattice where, via a coupling to Brownian directed percolation, it has been shown, in [8], to converge with proper renormalization to the Tracy-Widom distribution. The interested reader will find in the recent comprehensive survey [3] further results on these topics.

In a related subject, i.e., the study of the length of the longest common subsequences (*LCSs*) in random words, exact fluctuations have also been longed for. It is well known that the study of *LCSs* can be viewed as a directed last passage site percolation problem with random but dependent weights. In [27], the variance of the length of *LCSs* is shown to be linear when the letters are drawn from a highly biased Bernoulli distribution. This method is further developed in [18] to show that the r -th moment of *LCSs* is of order $\Theta(n^{r/2})$ under a similarly concentrated distribution, over a finite dictionary. This power lower bound on the fluctuations is essential in proving a Gaussian limiting law for the length of *LCSs*. (See

[16])

This chapter aims at studying the r -th, $1 \leq r < +\infty$, central moments of $DLPP$ in a thin rectangular $n \times \lfloor n^\alpha \rfloor$ -grid. They are shown to be lower-bounded by $n^{r(1-\alpha)/2}$, for $0 < \alpha < 1/3$, when n is large enough. (For $r = 1$, results on the first order central moments are very sparse in the percolation literature.) Moreover, our methodology is robust enough to also be applicable to first passage time in directed site/edge percolation.

Hereafter, for convenience, n^a will be short for $\lfloor n^a \rfloor$, $a > 0$. Next, the model under study is specified as follows: we consider a $n \times n^\alpha$ -grid having $n^{1+\alpha}$ vertices, each of which is associated with i.i.d. random weights w . The weight distribution is required to be non-degenerate and to have finite non-negative support, i.e., its *c.d.f.* F is such that $F(0-) = 0$ and such that there exists $C > 0$ with $F(C) = 1$. In this setting, the last passage time L_n is the maximum of the sums over all the weights, along all the unit-step up-right paths on the grid, from $(1, 1)$ to (n, n^α) . Namely,

$$L_n := \max_{\pi \in \Pi} \sum_{v \in \pi} w(v),$$

where Π is the set of all unit-step up-right paths from $(1, 1)$ to (n, n^α) , and where any path $\pi \in \Pi$ is an ordered set of vertices, i.e., $\pi = \{v_1 = (1, 1), v_2, \dots, v_{n+n^\alpha-1} = (n, n^\alpha)\}$ such that $v_{i+1} - v_i$, $i \in [n_1 + n_2 - 1] := \{1, 2, \dots, n_1 + n_2 - 1\}$, is either $\mathbf{e}_1 := (1, 0)$ or $\mathbf{e}_2 := (0, 1)$ and where $w : v \in [n] \times [n^\alpha] \rightarrow w(v) \in \mathbb{R}$ is the random weight associated with the vertex $v \in [n] \times [n^\alpha]$, where $[n] := \{1, 2, \dots, n\}$. Hereafter, *directed path* is short for such type of path. Further, any directed path realizing the last passage time is called a *geodesic*. Within this framework, our main result is as follows:

Theorem 2.1.1. *The r -th central moment of the directed last passage time in site percolation over a $n \times n^\alpha$ -grid, $0 < \alpha < 1/3$, is lower-bounded of order $n^{r(1-\alpha)/2}$, i.e., for $1 \leq r < +\infty$,*

$$\mathbb{M}_r(L_n) := \mathbb{E}(|L_n - \mathbb{E}L_n|^r) \geq c_0 n^{\frac{r(1-\alpha)}{2}},$$

where $c_0 > 0$ is a constant which depends on r but is independent of n .

Above, with the help of Hölder inequality and up to some worse constant, only the case $r = 1$ needs to be proved.

The remaining of this chapter is dedicated to the proof of the above theorem and is organized as follows: at the beginning of the next section, we show that with high probability the number of *hi*-mode weights (to be defined) on any geodesic grows at most linearly in n . More importantly, this indicates that there exist at least linearly many *lo*-mode weights on any geodesic. In turn, this helps showing that if L_n is represented as a random function of the number of *lo*-mode weights over the grid, then with high probability this function locally satisfies a reversed Lipschitz condition. In Section 2.3, the proof of the main theorem is completed by showing how such a local and reversed Lipschitz condition ensures the validity of a power lower bound on any central moment. In the concluding section, we briefly discuss the potential extension of our proof to the case of the second order central moment, i.e., the variance, over a square grid, i.e., $\alpha = 1$, and various related problems.

2.2 Preliminaries

We start by introducing the notions of *hi/lo*-mode of site weights: since the weight distribution is non-degenerate and non-negative, there exists $m > 0$ such that $\mathbb{P}(w > m) = p > 0$ and $\mathbb{P}(w \leq m) = 1 - p > 0$. Then, w is said to be in *hi*-mode if $w > m$; otherwise, w is in *lo*-mode. In addition, denote by M_n be the maximum of the number of weights in *hi*-mode over all directed paths, i.e.,

$$M_n := \max_{v \in \Pi} \sum_{v \in \pi} \mathbf{1}(w(v) > m),$$

which is nothing but the last passage time for the same grid with Bernoulli weights $\mathbf{1}(w(v) > m)$. In this section, L_n is considered as a function of the number of *hi*-mode weights over the grid, and is shown to locally satisfy a reversed Lipschitz condition, on an explicitly constructed event having very high probability.

2.2.1 Linear Growth of M_n

To begin, we show that there exists an absolute constant $0 < c_1 < 1$ such that the probability that M_n , i.e., the maximal number of *hi*-mode weights on any directed path, is larger than $c_1 n$ is exponentially small.

Proposition 2.2.1. *There exist constants $0 < c_1 < 1$ and $0 < c_2 < +\infty$, independent of n , such that*

$$\mathbb{P}(M_n \geq c_1 n) \leq \exp(-c_2 n), \quad (2.2.1)$$

for n large enough.

To prove Proposition 2.2.1, we start by showing a concentration inequality for M_n . This is achieved via the entropy method, and is akin to the proof of Theorem 3.12 presented in [3].

Proposition 2.2.2. *There exist constants $0 < c_3, c_4 < +\infty$, independent of n , such that for $t \in (0, c_4 \sqrt{n + n^\alpha - 1})$,*

$$\mathbb{P}(M_n - \mathbb{E}M_n \geq t\sqrt{n + n^\alpha - 1}) \leq \exp(-c_3 t^2).$$

Proof. Let $\psi(\lambda) = \log \mathbb{E} \exp(\lambda(M_n - \mathbb{E}M_n))$. Then, as shown next, it suffices to show that for some $c > 0$ and $\lambda \in (0, c)$,

$$\psi(\lambda) \leq c(n + n^\alpha - 1)\lambda^2. \quad (2.2.2)$$

Indeed, for any $\lambda > 0$,

$$\begin{aligned} \mathbb{P}(M_n - \mathbb{E}M_n \geq \sqrt{n + n^\alpha - 1}t) &\leq \mathbb{P}(\exp(\lambda(M_n - \mathbb{E}M_n)) \geq \exp(t\lambda\sqrt{n + n^\alpha - 1})) \\ &\leq \exp(\psi(\lambda) - t\lambda\sqrt{n + n^\alpha - 1}) \\ &\leq \exp(c(n + n^\alpha - 1)\lambda^2 - t\lambda\sqrt{n + n^\alpha - 1}). \end{aligned}$$

Letting $\lambda = t\sqrt{n + n^\alpha - 1}/2c$ will complete the proof, whenever (2.2.2), which we proceed to prove next, holds true. For any non-negative random variable X (and the convention

($0 \ln 0 = 0$), let as usual, $Ent X = \mathbb{E} X \log X - \mathbb{E} X \log \mathbb{E} X$. Then,

$$\begin{aligned}
\frac{d}{d\lambda} \left(\frac{\psi(\lambda)}{\lambda} \right) &= \frac{d}{d\lambda} \left(\frac{1}{\lambda} \ln \mathbb{E} \exp(\lambda(M_n - \mathbb{E} M_n)) \right) \\
&= -\frac{1}{\lambda^2} \ln \mathbb{E} \exp(\lambda(M_n - \mathbb{E} M_n)) + \frac{1}{\lambda} \frac{\mathbb{E}(M_n - \mathbb{E} M_n) \exp(\lambda(M_n - \mathbb{E} M_n))}{\mathbb{E} \exp(\lambda(M_n - \mathbb{E} M_n))} \\
&= -\frac{1}{\lambda^2} (\ln \mathbb{E} \exp(\lambda M_n) - \lambda \mathbb{E} M_n) + \frac{\mathbb{E}(M_n - \mathbb{E} M_n) \exp(\lambda M_n)}{\lambda \mathbb{E} \exp(\lambda M_n)} \\
&= \frac{\mathbb{E} M_n}{\lambda} - \frac{1}{\lambda^2} \ln \mathbb{E} \exp(\lambda M_n) + \frac{\mathbb{E} L \exp(\lambda M_n)}{\lambda \mathbb{E} \exp(\lambda M_n)} - \frac{\mathbb{E} M_n}{\lambda} \\
&= \frac{\lambda \mathbb{E} M \exp(\lambda M_n) - \mathbb{E} \exp(\lambda M_n) \ln \mathbb{E} \exp(\lambda M_n)}{\lambda^2 \mathbb{E} \exp(\lambda M_n)} \\
&= \frac{Ent \exp(\lambda M_n)}{\lambda^2 \mathbb{E} \exp(\lambda M_n)}.
\end{aligned}$$

If

$$Ent \exp(\lambda M_n) \leq c(n + n^\alpha - 1) \lambda^2 \mathbb{E} \exp(\lambda M_n), \quad (2.2.3)$$

for $\lambda \in (0, c)$, then we would have

$$\frac{d}{d\lambda} \left(\frac{\psi(\lambda)}{\lambda} \right) = \frac{Ent \exp(\lambda M_n)}{\lambda^2 \mathbb{E} \exp(\lambda M_n)} \leq c(n + n^\alpha - 1),$$

from which, it would follow that $\psi(\lambda) \leq c(n + n^\alpha - 1) \lambda^2$. Let us therefore prove (2.2.3).

First, enumerate the $n^{1+\alpha}$ vertices as $v_1, v_2, \dots, v_{n^{1+\alpha}}$ and denote the associated Bernoulli weights as $w(v_i)$. Now, recall the symmetrized modified logarithmic Sobolev inequality (see [9, Theorem 6.15]): for all $t \in \mathbb{R}$ and $Z = f(X_1, \dots, X_k)$, where X_1, X_2, \dots, X_k , $k \geq 1$, are independent and where f is a Borel function for which all the expectations below do exist,

$$Ent \exp(tZ) \leq \sum_{i=1}^k \mathbb{E} \left(\exp(tZ) q(-t(Z - Z'_i)_+) \right), \quad (2.2.4)$$

with $q(x) = x(e^x - 1)$, and with $Z'_i = f(X_1, \dots, X'_i, \dots, X_k)$, where X'_i is an independent copy of X_i . (This can be proved combining the usual tensorization property of the entropy with its variational representation.) Therefore, (2.2.4) leads to

$$Ent \exp(\lambda M) \leq \sum_{i=1}^{n^{1+\alpha}} \mathbb{E} \left(\exp(\lambda M) q(-\lambda(M - M'_i)_+) \right), \quad (2.2.5)$$

where now M is short for M_n and M is changed into M'_i when only the weight $w(v_i)$ is resampled as $w'(v_i)$, independently from $(w(v_i))_{i=1}^{n^{1+\alpha}}$, e.g., when $(w(v_i))_{i=1}^{n^{1+\alpha}}$ and $(w'(v_i))_{i=1}^{n^{1+\alpha}}$

are independent copy of each other. However, it is clear that $M - M'_i \leq 1$ with equality if and only if $w(v_i) = 1$ and, its independent copy, $w'(v_i) = 0$, for $v_i \in \mathcal{G}$, where \mathcal{G} is the set of vertices in the intersection of all the geodesics, i.e., $\mathcal{G} = \cap_{geodesics} \{v \in geodesic\}$. So it follows that

$$(M - M'_i)_+ \leq 1 - w'(v_i),$$

which in turn yields that

$$-\lambda(M - M'_i)_+ \geq -\lambda(1 - w'(v_i)).$$

On the other hand, $q'(x) = xe^x + e^x - 1 < 0$, when $x < 0$, and therefore

$$q(-\lambda(M - M'_i)_+) \leq q(-\lambda(1 - w'(v_i))).$$

Moreover, $q(0) = 0$ gives us

$$\exp(\lambda M)q(-\lambda(M - M'_i)_+) = \exp(\lambda M)q(-\lambda(M - M'_i)_+)\mathbf{1}(v_i \in \mathcal{G}).$$

Thus, via (2.2.5),

$$\begin{aligned} Ent \exp(\lambda M) &\leq \sum_{i=1}^{n^{1+\alpha}} \mathbb{E} \left(\exp(\lambda M)q(-\lambda(M - M'_i)_+)\mathbf{1}(v_i \in \mathcal{G}) \right) \\ &\leq \sum_{i=1}^{n^{1+\alpha}} \mathbb{E} \left(\exp(\lambda M)q(-\lambda(1 - w'(v_i)))\mathbf{1}(v_i \in \mathcal{G}) \right) \\ &= \sum_{i=1}^{n^{1+\alpha}} \mathbb{E} \left(\exp(\lambda M)\mathbf{1}(v_i \in \mathcal{G}) \right) \mathbb{E} \left(q(-\lambda(1 - w'(v_i))) \right) \\ &= \mathbb{E} \left(\exp(\lambda M)Card(\mathcal{G}) \right) \mathbb{E} \left(q(-\lambda(1 - w'(v_1))) \right). \end{aligned}$$

Since any geodesic covers exactly $n + n^\alpha - 1$ vertices, $Card(\mathcal{G}) \leq n + n^\alpha - 1$, and

$$Ent \exp(\lambda M) \leq (n + n^\alpha - 1) \mathbb{E} q(-\lambda(1 - w(v_1))) \mathbb{E} \exp(\lambda M). \quad (2.2.6)$$

Now, by dominated convergence,

$$\begin{aligned} \lim_{\lambda \searrow 0} \frac{\mathbb{E} q(-\lambda(1 - w(v_1)))}{\lambda^2} &= \mathbb{E} \left(\lim_{\lambda \searrow 0} \frac{(1 - w(v))(1 - \exp(-\lambda(1 - w(v))))}{\lambda} \right) \\ &= \mathbb{E}(1 - w(v_1))^2 = 1 - p. \end{aligned} \quad (2.2.7)$$

Hence, there exists c such that when $\lambda \in (0, c)$, $\mathbb{E}q(-\lambda(1 - w(v_1))) \leq \lambda^2$. Combining (2.2.6) with (2.2.7), it finally follows that

$$\text{Ent exp}(\lambda M) \leq (n + n^\alpha - 1)\lambda^2 \mathbb{E} \exp(\lambda M),$$

for $\lambda \in (0, c)$. □

Remark 2.2.3. *Note that in Proposition 2.2.2, and in contrast to [12, Theorem 1.1], the subcritical condition, i.e., $p < p_c$, where p_c is the critical probability in directed bond percolation, in two dimensions, is not required. This is mainly due to the fact that the subcritical condition is needed there to bound the length of the geodesics in undirected percolation; however, in our directed case, any directed path is naturally of length $n + n^\alpha - 1$.*

Proof of Proposition 2.2.1: Let g be the shape function, i.e., let $g((1, a)) = \lim_{n \rightarrow +\infty} \mathbb{E}M(n, na)/n$, where $M(n, na)$ is the last passage time over a $n \times \lfloor na \rfloor$ grid. It is shown in [29] that $g((1, a)) = p + 2\sqrt{p(1-p)a} + o(\sqrt{a})$, as $a \rightarrow 0$. Hence, for all n large enough, $\mathbb{E}M(n, n^\alpha) \leq (p+1)n/2$, which, when combined with Proposition 2.2.2, gives $\mathbb{P}(M_n \geq (p+1)n/2 + t\sqrt{n+n^\alpha-1}) \leq \exp(-c_1 t^2)$, for any $t \in (0, c_4\sqrt{n+n^\alpha-1})$.

Further, let $0 < \varepsilon < (1-p)/2$. Then there exists a constant $0 < \varepsilon_1 < c_4$, independent of n , such that if $t = \varepsilon_1\sqrt{n+n^\alpha-1} \in (0, c_4\sqrt{n+n^\alpha-1})$, then $t\sqrt{n+n^\alpha-1} \leq \varepsilon n$ and $t^2 = \varepsilon_1^2(n+n^\alpha-1) > \varepsilon_1 n$. Hence, for this particular t , $\mathbb{P}(M_n \geq (\varepsilon + (p+1)/2)n) \leq \exp(-c_3 \varepsilon_1^2 n)$. Setting $c_1 = \varepsilon + (p+1)/2 < 1$ and $c_2 = c_3 \varepsilon_1^2 > 0$, finishes the proof. □

2.2.2 Local Reversed Lipschitz Condition

To begin with, let us set the underlying probability space as $\Omega_n = \mathbb{R}^{n^{1+\alpha}}$ associated with the product measure $\bigotimes_{i=1}^{n^{1+\alpha}} F$ and let $W = (w(v_i))_{i=1}^{n^{1+\alpha}}$ be the random vector of weights under an arbitrary but deterministic enumeration of weights over all the $n^{1+\alpha}$ vertices. Let N be the total number of v_i such that $w(v_i)$ is in *hi*-mode and so, clearly, N is a binomial variable with parameters $n^{1+\alpha}$ and p . In addition, any weight w can be determined in a two-step way: it is first fixed to be in *hi/lo*-mode by flipping a Bernoulli random variable with parameter p ; then it is further associated with a non-negative weight by drawing from F conditional on the fixed *hi/lo*-mode of the first step. Based on this point of view, one can

construct an iterative scheme to sample W by starting from a grid with all the weights in *lo*-mode and changing a (random) binomial amount of them into *hi*-mode one at a time.

To be more precise, a (finite) sequence of random vectors of weights $\{W^k = (w^k(v_i))_{i=1}^{n^{1+\alpha}}\}_{k=0}^{n^{1+\alpha}}$ is iteratively defined as follows: First, let $W^0 = \{w^0(v_i)\}_{i=1}^{n^{1+\alpha}}$, where $w^0(v_i)$ has distribution F conditional on being in *lo*-mode. Thus, W^0 is clearly identical, in distribution, to W conditioned on $N = 0$. Second, once W^k , $k \geq 0$, is defined, one vertex v_{i_0} is uniformly chosen at random from the set $\{v_i : w^k(v_i) \text{ in } lo\text{-mode}\}$ and then W^{k+1} is defined in such a way that $w^{k+1}(v_{i_0})$ is sampled from F conditional on being in *hi*-mode while $w^{k+1}(v_i) = w^k(v_i)$, for $i \neq i_0$, i.e., W^{k+1} is defined by changing one uniformly chosen *lo*-mode weight in W^k to a *hi*-mode weight. The second step is repeated $n^{1+\alpha}$ times until all original *lo*-mode weights, in W^0 , are changed into only *hi*-mode weights, in $W^{n^{1+\alpha}}$.

From its very definition, for $0 \leq k \leq n^{1+\alpha}$, W^k contains k *hi*-mode weights. Moreover, $\{W^k\}_{k=0}^{n^{1+\alpha}}$ are dependent random variables but independent of both W and N . Next, we show that W^k has the same law as W conditioned on $N = k$.

Lemma 2.2.4. *For any $k = 0, 1, \dots, n^{1+\alpha}$,*

$$W^k =_d (W \mid N = k), \quad (2.2.8)$$

and moreover,

$$W^N =_d W, \quad (2.2.9)$$

where $=_d$ denotes equality in distribution.

Proof. The proof is by induction on k . By definition, $W^0 =_d W$ conditioned on $N = 0$. Assume now that (2.2.8) is true for $k \geq 1$, i.e., that for any $(\omega_i)_{i=1}^{n^{1+\alpha}} \in \Omega_n$ such that $\text{Card}(\{\omega_i \text{ in } lo\text{-mode}\}_{i=1}^{n^{1+\alpha}}) = k$,

$$\mathbb{P}(W^k = (\omega_i)_{i=1}^{n^{1+\alpha}}) = \binom{n^{1+\alpha}}{k}^{-1}. \quad (2.2.10)$$

Then, for any $(\omega_i)_{i=1}^{n^{1+\alpha}} \in \Omega_n$ such that $\text{Card}(\{\omega_i \text{ in } lo\text{-mode}\}_{i=1}^{n^{1+\alpha}}) = k + 1$,

$$\mathbb{P}(W^{k+1} = (\omega_i)_{i=1}^{n^{1+\alpha}}) = \sum_{j=1}^{k+1} \mathbb{P}(W^{k+1} = (\omega_i)_{i=1}^{n^{1+\alpha}} \mid B_j^{k+1}) \mathbb{P}(B_j^{k+1}), \quad (2.2.11)$$

where B_j^{k+1} , $1 \leq j \leq k+1$, denotes the event that the j th weight 1 in $\{\omega_i^{k+1} : \omega_i^{k+1} = 1, 1 \leq i \leq n^{1+\alpha}\}$ is the one which has been flipped uniformly at random from the weight 0 in W^k . Combining (2.2.10) and (2.2.11) gives

$$\begin{aligned} \mathbb{P}\left(W^{k+1} = (\omega_i)_{i=1}^{n^{1+\alpha}}\right) &= \sum_{j=1}^{k+1} \binom{n^{1+\alpha}}{k}^{-1} \frac{1}{n^{1+\alpha} - k} \\ &= \frac{k! (n^{1+\alpha} - k)}{(n^{1+\alpha})!} \frac{k+1}{n^{1+\alpha} - k} \\ &= \binom{n^{1+\alpha}}{k+1}^{-1}. \end{aligned}$$

Next, (2.2.8) and the independence of N and $\{W^k\}_{k=0}^{n^{1+\alpha}}$ give

$$\begin{aligned} \mathbb{E}(\exp(i\langle t, W \rangle)) &= \sum_{k=0}^{n^{1+\alpha}} \mathbb{E}(\exp(i\langle t, W \rangle) | N = k) \mathbb{P}(N = k) \\ &= \sum_{k=0}^{n^{1+\alpha}} \mathbb{E}(\exp(i\langle t, W^k \rangle) | N = k) \mathbb{P}(N = k) \\ &= \sum_{k=0}^{n^{1+\alpha}} \mathbb{E}(\exp(i\langle t, W^N \rangle) | N = k) \mathbb{P}(N = k) \\ &= \mathbb{E}(\exp(i\langle t, W^N \rangle)). \end{aligned}$$

□

This particular way of iterative sampling provides a new point of view on L_n . Letting $L_n(k) := L_n(W^k)$ and $L_n := L_n(W)$ respectively be the last passage times under the weights settings W^k and W , it is clear from Lemma 2.2.4, that $L_n(N) =_d L_n$ and so it is equivalent to study $\mathbb{M}_r(L_n(N))$ or $\mathbb{M}_r(L_n)$. We finish this section by showing that on an event of probability exponentially close to 1, $\{L_n(k)\}_{k=1}^{n^{1+\alpha}}$ locally satisfies a reversed Lipschitz condition. The beginning of the proof of this result is akin to a corresponding proof in [16].

Lemma 2.2.5. *There exist constants $0 < c_2, c_5, c_6 < +\infty$ independent of n such that, when*

n is large enough,

$$\begin{aligned} & \mathbb{P} \left(O_n := \bigcap_{i, j \in I, j \geq i + c_6 \sqrt{p(1-p)n^{1+\alpha}}} \left\{ L_n(j) - L_n(i) \geq \frac{c_5}{2n^\alpha} (j - i) \right\} \right) \\ & \geq 1 - 12p(1-p)n^{1+\alpha} \exp(-c_2 n) - p(1-p)n^{1+\alpha} \exp \left(-\frac{c_5^2 c_6 \sqrt{p(1-p)}}{4} n^{\frac{1-3\alpha}{2}} \right), \end{aligned}$$

where $I = [n^{1+\alpha}p - \sqrt{(1-p)pn^{1+\alpha}}, n^{1+\alpha}p + \sqrt{(1-p)pn^{1+\alpha}}]$.

Proof. Define a set $B_n = \{\omega : \omega \in \Omega_n, M_n(\omega) < c_1 n\}$ and so, by Proposition 2.2.1, $\mathbb{P}(B_n) \geq 1 - \exp(-c_2 n)$, when n is large enough. Further, let $A_n := \{W \in B_n\}$ and let $A_n^k := \{W^k \in B_n\}$. Then, by Lemma 2.2.4,

$$\mathbb{P} \left(\left(\bigcap_{k \in I} A_n^k \right)^c \right) \leq \sum_{k \in I} \mathbb{P} \left((A_n^k)^c \right) = \sum_{k \in I} \mathbb{P}(A_n^c \mid N = k) \leq \sum_{k \in I} \frac{\mathbb{P}(A_n^c)}{\mathbb{P}(N = k)}. \quad (2.2.12)$$

Meanwhile, for any $k \in I$, $\mathbb{P}(N = k) \geq 1/(6\sqrt{n^{1+\alpha}p(1-p)})$. Indeed,

$$\begin{aligned} & \mathbb{P}(N = k) \\ & \geq \min \left(\mathbb{P} \left(N = pn^{1+\alpha} - \lfloor \sqrt{(1-p)pn^{1+\alpha}} \rfloor \right), \mathbb{P} \left(N = pn^{1+\alpha} + \lfloor \sqrt{(1-p)pn^{1+\alpha}} \rfloor \right) \right), \end{aligned}$$

and, by de Moivre–Laplace Theorem,

$$\begin{aligned} \mathbb{P}(N = pn^{1+\alpha} - \lfloor \sqrt{(1-p)pn^{1+\alpha}} \rfloor) & \geq \frac{1}{2\sqrt{n^{1+\alpha}p(1-p)}} \exp \left(-\frac{\left(\lfloor \sqrt{(1-p)pn^{1+\alpha}} \rfloor \right)^2}{(1-p)pn^{1+\alpha}} \right) \\ & \geq \frac{1}{6\sqrt{n^{1+\alpha}p(1-p)}}, \end{aligned}$$

when n is large enough. Similarly, this lower bound also holds for $\mathbb{P}(N = pn^{1+\alpha} + \lfloor \sqrt{(1-p)pn^{1+\alpha}} \rfloor)$ and therefore

$$\mathbb{P}(N = k) \geq \frac{1}{6\sqrt{n^{1+\alpha}p(1-p)}}, \quad (2.2.13)$$

for any $k \in I$. Combining (2.2.1), (2.2.12) and (2.2.13) gives:

$$\begin{aligned} \mathbb{P} \left(\left(\bigcap_{k \in I} A_n^k \right)^c \right) & \leq 2\sqrt{(1-p)pn^{1+\alpha}} 6\sqrt{n^{1+\alpha}p(1-p)} \mathbb{P}(A_n^c) \\ & \leq 12p(1-p)n^{1+\alpha} \exp(-c_2 n). \end{aligned} \quad (2.2.14)$$

We show next that, with high probability, the difference between $L_n(k+1)$ and $L_n(k)$, conditioned on W^k , can be lower bounded by a fractional polynomial in n . Indeed, if $\mathbb{E}(w|hi)$ denotes the expectation of w conditioned on being in hi -mode, then

$$\mathbb{E}\left(L_n(k+1) - L_n(k)|W^k\right) \geq \frac{n + n^\alpha - M_n(k)}{n^{1+\alpha} - k} (\mathbb{E}(w|hi) - m),$$

since $L_n(k+1)$ increases if and only if the chosen lo -mode weight is on any geodesic under W^k . Next, note that there are at least many $n + n^\alpha - M_n(k)$ lo -mode weights on any geodesic and $n^{1+\alpha} - k$ many lo -mode weights over the grid under W^k , and so the probability that any lo -mode weight on some geodesic is chosen is at least $(n + n^\alpha - M_n(k)) / (n^{1+\alpha} - k)$. In addition, the expected increment of a single flipping should be $(\mathbb{E}(w|hi) - m) > 0$. Hence, by conditioning on $A_n^k = \{M_n(k) < c_1 n\}$,

$$\mathbb{E}\left(L_n(k+1) - L_n(k)|W^k\right) \geq \frac{(1 - c_1)}{n^\alpha} (\mathbb{E}(w|hi) - m). \quad (2.2.15)$$

Based on this lower bound, a martingale difference sequence is built as follows: for each $k \geq 0$, let

$$\Delta_{k+1} = \begin{cases} L_n(k+1) - L_n(k), & \text{if } A_n^k \text{ holds,} \\ (1 - c_1) (\mathbb{E}(w|hi) - m) / n^\alpha & \text{otherwise.} \end{cases}$$

Then, letting $c_5 := (1 - c_1) (\mathbb{E}(w|hi) - m)$,

$$\mathbb{E}\left(\Delta_{k+1}|W^k\right) \geq \frac{c_5}{n^\alpha}. \quad (2.2.16)$$

Now, for each $k = 0, 1, \dots, n^{1+\alpha}$, let $\mathcal{F}_k := \sigma(W^0, W^1, \dots, W^k)$, be the σ -field generated by W^0, W^1, \dots, W^k . Clearly, $\{\Delta_k - \mathbb{E}(\Delta_k|\mathcal{F}_{k-1}), \mathcal{F}_k\}_{1 \leq k \leq n^{1+\alpha}}$ forms a martingale differences sequence and since $0 \leq \Delta_k \leq C$ and thus $-C \leq \Delta_k - \mathbb{E}(\Delta_k|\mathcal{F}_{k-1}) \leq C$, Hoeffding's exponential martingale inequality gives, for any $i < j$,

$$\begin{aligned} \mathbb{P}\left(\sum_{k=i+1}^j (\Delta_k - \mathbb{E}(\Delta_k|\mathcal{F}_{k-1})) < -\frac{c_5}{2n^\alpha}(j-i)\right) &\leq \exp\left(-\frac{2c_5^2(j-i)^2}{4n^{2\alpha} \sum_{k=i+1}^j C^2}\right) \\ &= \exp\left(-\frac{c_5^2(j-i)}{2n^{2\alpha} C^2}\right). \end{aligned}$$

Moreover, from (2.2.16),

$$\sum_{k=i+1}^j \mathbb{E}\left(\Delta_k|W^{k-1}\right) \geq \frac{c_5}{n^\alpha}(j-i),$$

and therefore,

$$\begin{aligned} \mathbb{P}\left(\sum_{k=i+1}^j \Delta_k \leq \frac{c_5}{2n^\alpha}(j-i)\right) &\leq \mathbb{P}\left(\sum_{k=i+1}^j (\Delta_k - \mathbb{E}(\Delta_k|\mathcal{F}_{k-1})) < -\frac{c_5}{2n^\alpha}(j-i)\right) \\ &\leq \exp\left(-\frac{c_5^2(j-i)}{2n^{2\alpha}C^2}\right). \end{aligned} \quad (2.2.17)$$

For each $n \geq 1$, set

$$O_n^\Delta = \bigcap_{i,j \in I, j \geq i+\ell(n)} \left\{ \sum_{k=i+1}^j \Delta_k \geq \frac{c_5}{2n^\alpha}(j-i) \right\},$$

where $\ell(n) \geq 0$ will be fixed later. Then, by (2.2.17),

$$\begin{aligned} \mathbb{P}((O_n^\Delta)^c) &\leq \sum_{i,j \in I, j \geq i+\ell(n)} \mathbb{P}\left(\sum_{k=i+1}^j \Delta_k < \frac{c_5}{2n^\alpha}(j-i)\right) \\ &\leq \text{Card}(I)^2 \exp\left(-\frac{c_5^2 \ell(n)}{2n^{2\alpha}C^2}\right) \\ &= p(1-p)n^{1+\alpha} \exp\left(-\frac{c_5^2 \ell(n)}{2n^{2\alpha}C^2}\right). \end{aligned} \quad (2.2.18)$$

Now, on the intersection of the A_n^k , $k \in [i, j]$, $\sum_{k=i+1}^j \Delta_k = L_n(j) - L_n(i)$. Further, by the very definitions of Δ_k and O_n ,

$$\left(\bigcap_{k \in I} A_n^k\right) \cap O_n^\Delta \subseteq O_n.$$

Therefore, combining (2.2.14) and (2.2.18) and letting $\ell(n) = c_6 \sqrt{p(1-p)n^{1+\alpha}}$ gives

$$\begin{aligned} \mathbb{P}((O_n)^c) &\leq \mathbb{P}\left(\left(\bigcap_{k \in I} A_n^k\right)^c\right) + \mathbb{P}((O_n^\Delta)^c) \\ &\leq 12p(1-p)n^{1+\alpha} \exp(-c_2 n) + p(1-p)n^{1+\alpha} \exp\left(-\frac{c_5^2 \ell(n)}{2n^{2\alpha}C^2}\right) \\ &= 12p(1-p)n^{1+\alpha} \exp(-c_2 n) + p(1-p)n^{1+\alpha} \exp\left(-\frac{c_5^2 c_6 \sqrt{p(1-p)}}{2C^2} n^{\frac{1-3\alpha}{2}}\right). \end{aligned} \quad (2.2.19)$$

Clearly, when $\alpha < 1/3$, the right hand side of (2.2.19) converges, to 0, exponentially fast, as $n \rightarrow +\infty$. \square

2.3 Proof of Theorem 2.1.1

The beginning of the proof is similar to a corresponding proof in [18]. For a random variable U with finite r -th moment and for a random vector V , let $\mathbb{M}_r(U|V) := \mathbb{E}(|U - \mathbb{E}(U|V)|^r|V)$.

Clearly, by convexity and the conditional Jensen's inequality,

$$\begin{aligned}\mathbb{M}_r(U|V) &\leq 2^r ((\mathbb{E}(|U - \mathbb{E}U|^r|V)) / 2 + \mathbb{E}(|\mathbb{E}(U|V) - \mathbb{E}U|^r|V) / 2) \\ &\leq 2^r \mathbb{E}(|U - \mathbb{E}U|^r|V),\end{aligned}\tag{2.3.1}$$

and so, for any $n \geq 1$,

$$\begin{aligned}\mathbb{M}_r(L_n(N)) &\geq \frac{1}{2^r} \mathbb{E} \left(\mathbb{M}_r \left(L_n(N) | (L_n(k))_{0 \leq k \leq n^{1+\alpha}} \right) \right) \\ &= \frac{1}{2^r} \int_{\Omega_n} \mathbb{M}_r \left(L_n(N) | (L_n(k))_{0 \leq k \leq n^{1+\alpha}}(\omega) \right) \bigotimes_{i=1}^{n^{1+\alpha}} F(d\omega) \\ &\geq \frac{1}{2^r} \int_{O_n} \mathbb{M}_r \left(L_n(N) | (L_n(k))_{0 \leq k \leq n^{1+\alpha}}(\omega) \right) \bigotimes_{i=1}^{n^{1+\alpha}} F(d\omega).\end{aligned}\tag{2.3.2}$$

Moreover, since N is independent of $(L_n(k))_{0 \leq k \leq n^{1+\alpha}}$, and from (2.3.1), for each $\omega \in \Omega_n$,

$$\begin{aligned}&\mathbb{M}_r \left(L_n(N) | (L_n(k))_{0 \leq k \leq n^{1+\alpha}}(\omega) \right) \\ &\geq \frac{1}{2^r} \mathbb{M}_r \left(L_n(N) | (L_n(k))_{0 \leq k \leq n^{1+\alpha}}(\omega), \mathbf{1}_{N \in I} = 1 \right) \mathbb{P} \left(N \in I | (L_n(k))_{0 \leq k \leq n^{1+\alpha}}(\omega) \right) \\ &= \frac{1}{2^r} \mathbb{M}_r \left(L_n(N) | (L_n(k))_{0 \leq k \leq n^{1+\alpha}}(\omega), \mathbf{1}_{N \in I} = 1 \right) \mathbb{P}(N \in I).\end{aligned}\tag{2.3.3}$$

In addition (see [18, Lemma 2.2]), if $f : D \rightarrow \mathbb{Z}$ locally satisfies a reversed Lipschitz condition, i.e., f is such that for any $i, j \in D$ with $j > i + \ell$, $\ell \geq 0$, $f(j) - f(i) \geq c(j - i)$ for some $c > 0$ and if T is any D -valued random variable such that $\mathbb{E}|f(T)|^r < +\infty$, $r \geq 1$, then

$$\mathbb{M}_r(f(T)) \geq \left(\frac{c}{2} \right)^r (\mathbb{M}_r(T) - \ell^r).$$

So, for each $\omega \in O_n$, since N is independent of $(L_n(k))_{0 \leq k \leq n^{1+\alpha}}$,

$$\mathbb{M}_r \left(L_n(N) | (L_n(k))_{0 \leq k \leq n^{1+\alpha}}(\omega), \mathbf{1}_{N \in I} = 1 \right) \geq \left(\frac{c_3}{n^\alpha} \right)^r (\mathbb{M}_r(N | \mathbf{1}_{N \in I} = 1) - \ell(n)^r).\tag{2.3.4}$$

Next, (2.3.2), (2.3.3) and (2.3.4) lead to

$$\mathbb{M}_r(L_n(N)) \geq \frac{c_3^r}{2^r n^{r\alpha}} (\mathbb{M}_r(N | \mathbf{1}_{N \in I} = 1) - \ell(n)^r) \mathbb{P}(N \in I) \mathbb{P}(O_n),\tag{2.3.5}$$

and it remains to estimate the first two terms on the right side of (2.3.5). By the Berry-Esséen Theorem, and for all $n \geq 1$,

$$\left| \mathbb{P}(N \in I) - \frac{1}{\sqrt{2\pi}} \int_{-1}^1 \exp\left(-\frac{x^2}{2}\right) dx \right| \leq \frac{1}{\sqrt{n^{1+\alpha} p(1-p)}}.\tag{2.3.6}$$

On the other hand,

$$\begin{aligned}
\mathbb{M}_r(N|\mathbf{1}_{N \in I} = 1) &= \mathbb{E}(|N - n^{1+\alpha}p + n^{1+\alpha}p - \mathbb{E}(N|\mathbf{1}_{N \in I} = 1)|^r | \mathbf{1}_{N \in I} = 1) \\
&\geq \left| \mathbb{E}(|N - n^{1+\alpha}p| | \mathbf{1}_{N \in I} = 1)^{1/r} - |n^{1+\alpha}p - \mathbb{E}(N|\mathbf{1}_{N \in I} = 1)| \right|^r,
\end{aligned} \tag{2.3.7}$$

and when n is large enough,

$$\begin{aligned}
&|n^{1+\alpha}p - \mathbb{E}(N|\mathbf{1}_{N \in I} = 1)| \\
&= \sqrt{n^{1+\alpha}p(1-p)} \left| \mathbb{E} \left(\frac{N - n^{1+\alpha}p}{\sqrt{n^{1+\alpha}p(1-p)}} | \mathbf{1}_{N \in I} = 1 \right) \right| \\
&= \sqrt{n^{1+\alpha}p(1-p)} \frac{|F_n(1) - \Phi(1) + F_n(-1) - \Phi(-1) - \int_{-1}^1 (F_n(x) - \Phi(x)) dx|}{\mathbb{P}(N \in I)} \\
&\leq \sqrt{n^{1+\alpha}p(1-p)} \frac{4 \max_{x \in [-1,1]} |F_n(x) - \Phi(x)|}{\mathbb{P}(N \in I)} \\
&\leq \sqrt{n^{1+\alpha}p(1-p)} \frac{2/\sqrt{n^{1+\alpha}p(1-p)}}{\int_{-1}^1 \exp\left(-\frac{x^2}{2}\right) dx / \sqrt{2\pi} - 1/\sqrt{n^{1+\alpha}p(1-p)}} \\
&\leq \frac{3}{\int_{-1}^1 \exp\left(-\frac{x^2}{2}\right) dx / \sqrt{2\pi}},
\end{aligned} \tag{2.3.8}$$

where F_n is the distribution function of $(N - n^{1+\alpha}p)/\sqrt{n^{1+\alpha}p(1-p)}$, while Φ is the standard normal one. Likewise,

$$\begin{aligned}
&\mathbb{E}(|N - n^{1+\alpha}p|^r | \mathbf{1}_{N \in I} = 1) \\
&\geq (n^{1+\alpha}p(1-p))^{r/2} \frac{\int_{-1}^1 |x|^r d\Phi(x) - 4 \max_{x \in [-1,1]} |F_n(x) - \Phi(x)|}{\mathbb{P}(N \in I)} \\
&\geq (n^{1+\alpha}p(1-p))^{r/2} \frac{\int_{-1}^1 |x|^r d\Phi(x) - 2\sqrt{\pi}/\sqrt{n^{1+\alpha}p(1-p)}}{\int_{-1}^1 \exp\left(-\frac{x^2}{2}\right) dx + \sqrt{\pi}/\sqrt{n^{1+\alpha}p(1-p)}} \\
&\geq (n^{1+\alpha}p(1-p))^{r/2} \frac{\int_{-1}^1 |x|^r d\Phi(x)}{2 \int_{-1}^1 \exp\left(-\frac{x^2}{2}\right) dx}.
\end{aligned} \tag{2.3.9}$$

Next, (2.3.7), (2.3.8) and (2.3.9) give

$$\begin{aligned}
&\mathbb{M}_r(N_1 | \mathbf{1}_{N_1 \in I} = 1) \\
&\geq n^{\frac{r(1+\alpha)}{2}} \left| \sqrt{p(1-p)} \left(\frac{\int_{-1}^1 |x|^r d\Phi(x)}{2 \int_{-1}^1 \exp\left(-\frac{x^2}{2}\right) dx} \right)^{1/r} - \frac{3}{\int_{-1}^1 \exp\left(-\frac{x^2}{2}\right) dx / \sqrt{2\pi}} \right|^r.
\end{aligned} \tag{2.3.10}$$

For $\mathbb{M}_r(N|\mathbf{1}_{N \in I} = 1)$ to dominate the first term $\mathbb{M}_r(N|\mathbf{1}_{N \in I} = 1) - \ell(n)^r$ in (2.3.5), the constant c_7 (which depends on r and p but not n) is chosen such that:

$$c_7(r, p) \leq \sqrt{p(1-p)} \left(\frac{\int_{-1}^1 |x|^r d\Phi(x)}{2 \int_{-1}^1 \exp\left(-\frac{x^2}{2}\right) dx} \right)^{1/r}.$$

Letting $\ell(n) = c_7 n^{(1+\alpha)/2}$, it follows that

$$\mathbb{M}_r(N|\mathbf{1}_{N \in I} = 1) - \ell(n)^r \geq n^{r(1+\alpha)/2} \left(\sqrt{p(1-p)} \left(\frac{\int_{-1}^1 |x|^r d\Phi(x)}{2 \int_{-1}^1 \exp\left(-\frac{x^2}{2}\right) dx} \right)^{1/r} - c_7 \right)^r.$$

This last estimate, combined with (2.3.6) and Lemma 2.2.5, finally gives

$$\begin{aligned} \mathbb{M}_r(L_n(N)) &\geq \frac{c_3^r}{2^r n^{r\alpha}} (\mathbb{M}_r(N|\mathbf{1}_{N \in I} = 1) - \ell(n)^r) \mathbb{P}(N \in I) \mathbb{P}(O_n) \\ &\geq \frac{c_3^r}{2^r n^{r\alpha}} \left(\frac{1}{2\sqrt{2\pi}} \int_{-1}^1 \exp\left(-\frac{x^2}{2}\right) dx \right) \\ &\quad n^{r(1+\alpha)/2} \left(\sqrt{p(1-p)} \left(\frac{\int_{-1}^1 |x|^r d\Phi(x)}{2 \int_{-1}^1 \exp\left(-\frac{x^2}{2}\right) dx} \right)^{1/r} - c_7 \right)^r \\ &\quad \left(1 - 12p(1-p)n^{1+\alpha} \exp(-c_2 n) + p(1-p)n^{1+\alpha} \exp\left(-\frac{c_5^2 c_6 \sqrt{p(1-p)}}{2C^2} n^{\frac{1-3\alpha}{2}}\right) \right) \\ &= c_0 n^{(1-\alpha)r/2}. \end{aligned}$$

2.4 Concluding Remarks

The major limitation of our method is the upper bound $1/3$ on α , which stems from an application of Hoeffding's classical exponential martingale inequality. Specifically, we note that there is some discrepancy between the orders of the upper and lower bounds for the martingale differences in (2.2.15) conditioned on the event O_n , i.e., the conditional lower bound is of order $o(n^{-\alpha})$ while the upper bound is of order $o(1)$. With this discrepancy, it takes exactly $\alpha < 1/3$ for the exponential concentration to hold. A more sophisticated way of flipping weights from *lo*-mode to *hi*-mode in the construction of the martingale might produce a way to mitigate this, so as to relieve the $1/3$ bound. A more powerful concentration inequality to replace Hoeffding's one could also be of use.

However, even if our method could be generalized to the case $\alpha = 1$, i.e., if the grid is perfect square, the corresponding lower bound for the variance will be $O(n^{1-\alpha=1}) = O(1)$

and thus of little use. Nevertheless, the fact that geodesics in $DLPP$ are confined to a cylinder, centered on the main diagonal of the grid, and of width of order strictly smaller than $o(n)$ will help producing a non-trivial lower bound. The typical order of the width of the cylinder is the transversal fluctuation, which is believed to be $n^{2/3}$. Further, it is also believed that with very high probability geodesics are confined to such a cylinder of width $o(n^{2/3+\epsilon})$, for $\epsilon > 0$. Actually it has been proved that the transversal fluctuation exponent can be upper bounded by $3/4$ in the setting of undirected first passage percolation in [30] and that exponential concentration holds for all the geodesics in a cylinder of width $O(n^{(2\kappa+2)/(2\kappa+3)}\sqrt{\ln n})$ in [20] in the current setting, both of which assume the finiteness of the curvature exponent $\kappa > 0$. This is equivalent to saying that if let \tilde{L}_n be the last passage time within the cylinder, then $\tilde{L}_n \geq L_n$ holds with exponentially high probability. So

$$\begin{aligned}\mathbb{E}\tilde{L}_n - \mathbb{E}L_n &= \mathbb{E}\left((\tilde{L}_n - L_n)\left(\mathbf{1}_{\{\tilde{L}_n \geq L_n\}} + \mathbf{1}_{\{\tilde{L}_n < L_n\}}\right)\right) \\ &= \mathbb{E}\left((\tilde{L}_n - L_n)\mathbf{1}_{\{\tilde{L}_n < L_n\}}\right) \geq -2n\mathbb{P}(\tilde{L}_n < L_n) \rightarrow 0,\end{aligned}$$

as $n \rightarrow +\infty$. Meanwhile, it is trivial that $\tilde{L}_n \leq L_n$. So, as $n \rightarrow +\infty$, $\mathbb{E}\tilde{L}_n - \mathbb{E}L_n \rightarrow 0$ exponentially fast. This shows the potential of bounding the variance of L_n by that of \tilde{L}_n . Indeed,

$$\begin{aligned}Var(L_n) &= Var(L_n - \mathbb{E}\tilde{L}_n) \\ &= \mathbb{E}\left(L_n - \mathbb{E}\tilde{L}_n\right)^2 - \left(\mathbb{E}L_n - \mathbb{E}\tilde{L}_n\right)^2 \\ &= \mathbb{E}\left((L_n - \mathbb{E}\tilde{L}_n)\left(\mathbf{1}_{\{\tilde{L}_n \geq L_n\}} + \mathbf{1}_{\{\tilde{L}_n < L_n\}}\right)\right) - \left(\mathbb{E}L_n - \mathbb{E}\tilde{L}_n\right)^2 \\ &= \mathbb{E}\left((\tilde{L}_n - \mathbb{E}\tilde{L}_n)^2 \mathbf{1}_{\{\tilde{L}_n \geq L_n\}}\right) + \mathbb{E}\left((L_n - \mathbb{E}\tilde{L}_n) \mathbf{1}_{\{\tilde{L}_n < L_n\}}\right) - \left(\mathbb{E}L_n - \mathbb{E}\tilde{L}_n\right)^2 \\ &= Var(\tilde{L}_n) + \mathbb{E}\left(\left((L_n - \mathbb{E}\tilde{L}_n)^2 - (\tilde{L}_n - \mathbb{E}\tilde{L}_n)^2\right) \mathbf{1}_{\{\tilde{L}_n < L_n\}}\right) - \left(\mathbb{E}L_n - \mathbb{E}\tilde{L}_n\right)^2 \\ &\geq Var(\tilde{L}_n) - 8n^2\mathbb{P}(\tilde{L}_n < L_n) - \left(\mathbb{E}L_n - \mathbb{E}\tilde{L}_n\right)^2.\end{aligned}$$

In a symmetric manner, it is also true that $Var L_n \leq Var \tilde{L}_n + 8n^2\mathbb{P}(\tilde{L}_n < L_n) - \left(\mathbb{E}L_n - \mathbb{E}\tilde{L}_n\right)^2$. So the variances of L_n and \tilde{L}_n share the same asymptotic order. On the other hand, our methodology also applies, with small modifications, to \tilde{L}_n , i.e., the last passage time in the cylinder of length $O(n)$ and width $O(n^\alpha)$. This produces a power asymptotic lower bound

$n^{1-\alpha}$ for $Var(\tilde{L}_n)$ and so for $Var(L_n)$. In the best case, if it can be proved that, with exponentially high probability, the geodesics in directed last passage site percolation over a $n \times n$ grid are confined to a cylinder of the width $n^{2/3+\epsilon}$, $\epsilon > 0$, the corresponding power lower bound for the longitudinal fluctuation will be $n^{1-2/3-\epsilon} = n^{1/3-\epsilon}$. Although this is still not the tight conjectured bound $n^{2/3}$, it still serves as a good power lower bound.

Chapter III

A NOTE ON THE EXPECTED LENGTH OF THE LONGEST COMMON SUBSEQUENCES OF TWO I.I.D. RANDOM PERMUTATIONS

3.1 Introduction

The length of the longest increasing subsequences (*LISs*) of a uniform random permutation $\sigma \in \mathcal{S}_n$ (where \mathcal{S}_n is the symmetric group) is well studied and we refer to the monograph [34] for precise results and a comprehensive bibliography on this subject. Recently, [17] showed that for two independent random permutations $\sigma_1, \sigma_2 \in \mathcal{S}_n$, and as long as σ_1 is uniformly distributed and regardless of the distribution of σ_2 , the length of the longest common subsequences (*LCSs*) of the two permutations is identical in law to the length of the *LISs* of σ_1 , i.e. $LCS(\sigma_1, \sigma_2) \stackrel{\mathcal{L}}{=} LIS(\sigma_1)$. This equality ensures, in particular, that when σ_1 and σ_2 are uniformly distributed, $\mathbb{E}LCS(\sigma_1, \sigma_2)$ is upper bounded by $2\sqrt{n}$, for any n , (see [32]) and asymptotically of order $2\sqrt{n}$ ([34]). It is then rather natural to study the behavior of $LCS(\sigma_1, \sigma_2)$, when σ_1 and σ_2 are i.i.d. but not necessarily uniform. In this respect, Bukh and Zhou raised, in [10], two issues which can be rephrased as follows:

Conjecture. *Let P be an arbitrary probability distribution on \mathcal{S}_n . Let σ_1 and σ_2 be two i.i.d. permutations sampled from P . Then $\mathbb{E}_P[LCS(\sigma_1, \sigma_2)] \geq \sqrt{n}$. It might even be true that the uniform distribution U on \mathcal{S}_n gives a minimizer.*

Below we prove the suboptimality of the uniform distribution by explicitly building a distribution having a smaller expectation. In the next section, before presenting and proving our main result, we give a few definitions and formalize this minimizing problem as a quadratic programming one. Section 3.3 further explore some properties of the spectrum of the coefficient matrix of our quadratic program. In the concluding section, a quick cubic root lower bound is given along with a few pointers for future research.

3.2 Main Results

We begin with a few notations. Throughout, σ and π are, respectively, used for random and deterministic permutations. By convention, $[n] := \{1, 2, 3, \dots, n\}$ and so $\{\pi_i\}_{i \in [n!]} = \mathcal{S}_n$ is a particular ordered enumeration of \mathcal{S}_n . (Some other orderings of \mathcal{S}_n will be given when necessary.) Next, a random permutation σ is said to be sampled from $P = (p_i)_{i \in [n!]}$, if $\mathbb{P}_P(\sigma = \pi_i) = p_i$. The uniform distribution is therefore $U = (1/n!)_{i \in [n!]}$ and, for simplification, it is denoted by $E/n!$, where $E = (1)_{i \in [n!]}$ is the n -tuple only made up of ones. When needed, a superscript will indicate the degree of the symmetric group we are studying, e.g., $\sigma^{(n)}$ and $P^{(n)}$ are respectively a random permutation and distribution from \mathcal{S}_n .

Let us now formalize the expectation as a quadratic form:

$$\begin{aligned} \mathbb{E}_P[LCS(\sigma_1, \sigma_2)] &= \sum_{i,j \in [n!]} p_i LCS(\pi_i, \pi_j) p_j \\ &= \sum_{i,j \in [n!]} p_i \ell_{ij} p_j = P^T L^{(n)} P, \end{aligned} \quad (3.2.1)$$

where $\ell_{ij} := LCS(\pi_i, \pi_j)$ and $L^{(n)} := \{\ell_{ij}\}_{(i,j) \in [n!] \times [n!]}$. It is clear that $\ell_{ij} = \ell_{ji}$ and that $\ell_{ii} = n$. A quick analysis of the cases $n = 2$ or 3 shows that both $L^{(2)}$ and $L^{(3)}$ are positive semi-definite. However, this property does not hold further:

Lemma 3.2.1. *For $n \geq 4$, the smallest eigenvalue $\lambda_1^{(n)}$ of $L^{(n)}$ is negative.*

Proof. Linear algebra gives $\lambda_1^{(2)} = 1$ and $\lambda_1^{(3)} = 0$. So to prove the result, it suffices to show that $\lambda_1^{(k+1)} < \lambda_1^{(k)}$, $k \geq 1$ and this is done by induction. The base case is true, since $\lambda_1^{(2)} = 1 > 0 = \lambda_1^{(3)}$. To reveal the connection between $L^{(k+1)}$ and $L^{(k)}$, the enumeration of \mathcal{S}_{k+1} is iteratively built on that of \mathcal{S}_k by inserting the new element $(k+1)$ into the permutations from \mathcal{S}_k in the following way: the enumeration of the $(k+1)!$ permutations is split into $(k+1)$ trunks of equal size $k!$. In the i th trunk, the new element $(k+1)$ is inserted behind the $(k+1-i)$ th digit in the permutation from \mathcal{S}_k . (For example, if \mathcal{S}_2 is enumerated as $\{[12], [21]\}$, then the enumeration of the first trunk in \mathcal{S}_3 is $\{[123], [213]\}$, the second is $\{[132], [231]\}$ and the third is $\{[312], [321]\}$. Then the overall enumeration for \mathcal{S}_3 is $\{[123], [213], [132], [231], [312], [321]\}$.)

Via this enumeration, the principal minor of size $k! \times k!$ is row and column indexed by the enumeration of the permutations $\{\pi_i^{(k)}\}_{i \in [k!]}$ from \mathcal{S}_k with $(k+1)$ as the last digit, i.e., $\{[\pi_i^{(k)}(k+1)]\}_{i \in [k!]} \subseteq \mathcal{S}_{k+1}$. Then the (i, j) entry of the submatrix is

$$LCS([\pi_i(k+1)], [\pi_j(k+1)]) = LCS(\pi_i, \pi_j) + 1,$$

since the last digit $(k+1)$ adds an extra element into the longest common subsequences. Hence, the $k! \times k!$ principal minor of $L^{(k+1)}$ is $L^{(k)} + E^{(k)}(E^{(k)})^T$, where $E^{(k)}$ is the vector of $\mathbb{R}^{k!}$ only made up of ones. Moreover, notice that the sum of the π_i -indexed row of $L^{(k)}$ is

$$\begin{aligned} \sum_{j \in [k!]} LCS(\pi_i, \pi_j) &= \sum_{j \in [k!]} LCS(id, \pi_i^{-1}\pi_j) \\ &= \sum_{j \in [k!]} LIS(\pi_i^{-1}\pi_j), \end{aligned}$$

since simultaneously relabeling π_i and π_j does not change the length of the LCS s and also since a particular relabeling to make π_i to be the identity permutation, which is equivalent to left composition by π_i^{-1} , is applied here. Further, any LCS of the identity permutation and of $\pi_i^{-1}\pi_j$ is a LIS of $\pi_i^{-1}\pi_j$ and vice versa. So the row sum is equal to

$$\sum_{j \in [k!]} LIS(\pi_i^{-1}\pi_j) = \sum_{\pi \in \mathcal{S}_k} LIS(\pi),$$

since left composition by π_i^{-1} is a bijection from \mathcal{S}_k to \mathcal{S}_k . This indicates that all the row sums of $L^{(k)}$ are equal. Hence, $E^{(k)}$ is actually a right eigenvector of $L^{(k)}$ and is associated with the row sum $\sum_{\pi \in \mathcal{S}_k} LIS(\pi) > 0$ as its eigenvalue, which is distinct from the smallest eigenvalue $\lambda_1^{(k)} \leq 0$.

On the other hand, since $L^{(k)}$ is symmetric, the eigenvectors $R_1^{(k)}$ and $E^{(k)}$ associated with the eigenvalues $\lambda_1^{(k)}$ and $\sum_{\pi \in \mathcal{S}_k} LIS(\pi)$ are orthogonal, i.e.,

$$(E^{(k)})^T R_1^{(k)} = 0. \quad (3.2.2)$$

Without loss of generality, let $R_1^{(k)}$ be a unit vector, then from (3.2.2),

$$\begin{aligned} \lambda_1^{(k)} &= (R_1^{(k)})^T L^{(k)} (R_1^{(k)}) \\ &= (R_1^{(k)})^T (L^{(k)} + E^{(k)}(E^{(k)})^T) R_1^{(k)}. \end{aligned} \quad (3.2.3)$$

As $L^{(k)} + E^{(k)}(E^{(k)})^T$ is the $k! \times k!$ principal minor of $L^{(k+1)}$, (3.2.3) becomes

$$\begin{bmatrix} R_1^{(k)} \\ 0 \end{bmatrix}^T L^{(k+1)} \begin{bmatrix} R_1^{(k)} \\ 0 \end{bmatrix} \geq \min_{R^T E=0, \|R\|=1} R^T L^{(k+1)} R = \lambda_1^{(k+1)}, \quad (3.2.4)$$

where $R_1^{(k)}$ is properly extended to $\begin{bmatrix} R_1^{(k)} \\ 0 \end{bmatrix} \in \mathbb{R}^{(k+1)!}$ and where the above inequality holds

true since $\begin{bmatrix} R_1^{(k)} \\ 0 \end{bmatrix}^T E^{(k+1)} = (R_1^{(k)})^T E^{(k)} = 0$ and $\left\| \begin{bmatrix} R_1^{(k)} \\ 0 \end{bmatrix} \right\| = \|R_1^{(k)}\| = 1$, where $\|\cdot\|$ denotes the corresponding Euclidean norm. Moreover, equality in (3.2.4) holds if and only if $\begin{bmatrix} R_1^{(k)} \\ 0 \end{bmatrix}$ is a eigenvector of $L^{(k+1)}$ associated with $\lambda_1^{(k+1)}$. We show next, by contradiction, that this cannot be the case. Indeed, assume that

$$L^{(k+1)} \begin{bmatrix} R_1^{(k)} \\ 0 \end{bmatrix} = \lambda_1^{(k+1)} \begin{bmatrix} R_1^{(k)} \\ 0 \end{bmatrix}. \quad (3.2.5)$$

Now, consider the $k! \times k!$ submatrix at the bottom-left corner of $L^{(k+1)}$, which is row-indexed by $\{[(k+1)\pi_i]\}_{i \in [k!]}$ and column-indexed by $\{[\pi_i(k+1)]\}_{i \in [k!]}$. Notice that the (i, j) -entry of this submatrix is

$$LCS([(k+1)\pi_i], [\pi_j(k+1)]) = LCS(\pi_i, \pi_j),$$

since $(k+1)$ can be in some LCS only if the length of this LCS is 1. So this submatrix is in fact equal to $L^{(k)}$. Further, the vector consisting of the bottom $k!$ elements on the left-hand-side of (3.2.5) is $L^{(k)} R_1^{(k)} = \lambda_1^{(k)} R_1^{(k)}$, which is a non-zero vector. However, on the right-hand-side, the corresponding bottom $k!$ elements of the vector $\begin{bmatrix} R_1^{(k)} \\ 0 \end{bmatrix}$ form the zero vector. This leads to a contradiction. So,

$$\lambda_1^{(2)} = 1 > 0 = \lambda_1^{(3)} > \lambda_1^{(4)} > \lambda_1^{(5)} \dots$$

□

The above result on the smallest negative eigenvalue, and its associated eigenvector, will help build a distribution on \mathcal{S}_n , for which the LCS s have a smaller expectation than for the uniform one.

Theorem 3.2.2. *Let σ_1 and σ_2 be two i.i.d. random permutations sampled from a distribution P on the symmetric group \mathcal{S}_n . Then, for $n \leq 3$, the uniform distribution U minimizes $\mathbb{E}_P[LCS(\sigma_1, \sigma_2)]$, while, for $n \geq 4$, U is sub-optimal.*

Proof. As we have seen in (3.2.1),

$$\begin{aligned}
\mathbb{E}_P[LCS(\sigma_1, \sigma_2)] &= P^T L P \\
&= (P - U)^T L (P - U) + 2P^T L U - U^T L U \\
&= (P - U)^T L (P - U) + 2U^T L U - U^T L U \\
&= (P - U)^T L (P - U) + U^T L U,
\end{aligned} \tag{3.2.6}$$

where $P^T L U = U^T L U$, since U is an eigenvector of L and $P^T U = 1$.

When $n = 2, 3$, $L^{(n)}$ is positive semi-definite and therefore $(P - U)^T L (P - U) \geq 0$. So, $P^T L P \geq U^T L U$.

However, when $n \geq 4$, by Lemma 3.2.1, the smallest eigenvalue $\lambda_1^{(n)}$ is strictly negative and the associated eigenvector $R_1^{(n)}$ is such that $U^T R_1^{(n)} = 0 = E^T R_1^{(n)}$. Hence, there exists a positive constant c such that $cR_1^{(n)} \succeq -1/n!$, where \succeq stands for componentwise inequality. Let P_0 be such that $P_0 - U = cR_1^{(n)}$, then it is immediate that

$$E^T P_0 = E^T (U + cR_1^{(n)}) = 1 + 0 = 1,$$

and that

$$P_0 = U + cR_1^{(n)} \succeq 0.$$

Therefore, P_0 is a well-defined distribution on \mathcal{S}_n . On the other hand, by (3.2.6), the expectation under P_0 is such that

$$\begin{aligned}
\mathbb{E}_{P_0}[LCS(\sigma_1, \sigma_2)] &= (P_0 - U)^T L (P_0 - U) + U^T L U \\
&= c^2 (R_1^{(n)})^T L (R_1^{(n)}) + U^T L U \\
&= c^2 \lambda_1^{(n)} + U^T L U \\
&< U^T L U.
\end{aligned} \tag{3.2.7}$$

However, the right-hand side of (3.2.7) is nothing but the expectation under the uniform distribution, namely, $\mathbb{E}_U[LCS(\sigma_1, \sigma_2)]$. \square

The existence of negative eigenvalues contributes to the above construction and to the corresponding counterexample. So, as a next step, properties of this smallest negative eigenvalue and of the spectrum of the coefficient matrix $L^{(n)}$ are explored.

3.3 Further Properties of $L^{(n)}$

As we have seen, the vector $E^{(n)}$ which is made up of only ones is an eigenvector associated with the eigenvalue $\sum_{\pi \in S_n} LIS(\pi)$. It is not hard to show that this eigenvalue is, in fact, the spectral radius of $L^{(n)}$.

Proposition 3.3.1. $\sum_{\pi \in S_k} LIS(\pi)$ is the spectral radius of $L^{(n)}$.

Proof. Without loss of generality, let (λ, R) be a pair of eigenvalue and corresponding eigenvector of $L^{(n)}$ such that $\max_{i \in [n!]} |r_i| = 1$, where $R = (r_1, \dots, r_{n!})^T$, and let i_0 be the index such that $|r_{i_0}| = 1$. Let us focus now on the i_0 th element of λR . Then, since $L^{(n)}R = \lambda R$,

$$\begin{aligned} |\lambda| &= |\lambda r_{i_0}| \\ &= \left| \sum_{j \in [n!]} LCS(\pi_{i_0}, \pi_j) r_j \right| \\ &\leq \sum_{j \in [n!]} LCS(\pi_{i_0}, \pi_j) \\ &= \sum_{j \in [n!]} LIS(\pi_{i_0}^{-1} \pi_j) \\ &= \sum_{\pi \in S_n} LIS(\pi), \end{aligned}$$

with equality if and only if all the r_j 's have the same sign and have absolute value equal to 1. □

This gives a trivial bound on the smallest negative value $\lambda_1^{(n)}$: namely, $\lambda_1^{(n)} \geq -\sum_{\pi \in S_n} LIS(\pi)$. Moreover, since the expectation of the longest increasing subsequence of a uniform random permutation is asymptotically $2\sqrt{n}$, this gives an asymptotic order of $-2n!\sqrt{n}$ for the lower bound. On the other hand, we are interested in an upper bound for $\lambda_1^{(n)}$. The next result shows that $\lambda_1^{(n)}$ decreases at least exponentially fast, in n .

Proposition 3.3.2. $\lambda_1^{(n)} \leq 2^{n-4} \lambda_1^{(4)} = -2^{n-3} < 0$.

Proof. This is proved by showing that $\lambda_1^{(n+1)} \leq 2\lambda_1^{(n)}$. As well known,

$$\lambda_1^{(n+1)} = \min_{E^T R=0} \frac{R^T L^{(n+1)} R}{R^T R}. \quad (3.3.1)$$

Let $\lambda_1^{(n)}$ be the smallest eigenvalues of $L^{(n)}$ and let $R^{(n)}$ be the corresponding eigenvector.

Then, in generating $L^{(n+1)}$ from $L^{(n)}$ as done in the proof of Lemma 3.2.1, the $n! \times n!$ principal minor of $L^{(n+1)}$ is $L^{(n)} + EE^T$, while its bottom-left $n! \times n!$ submatrix is $L^{(n)}$.

Symmetrically, it can be proved that the top-right $n! \times n!$ submatrix is also $L^{(n)}$, while the bottom-right $n! \times n!$ submatrix is $L^{(n)} + EE^T$, i.e., $L^{(n+1)}$ is

$$\begin{bmatrix} L^{(n)} + EE^T & \cdots & L^{(n)} \\ \vdots & \ddots & \vdots \\ L^{(n)} & \cdots & L^{(n)} + EE^T \end{bmatrix}.$$

Further, let

$$R = \begin{bmatrix} R_1^{(n)} \\ 0 \\ \vdots \\ 0 \\ R_1^{(n)} \end{bmatrix}.$$

Then $E^T R = E^T R_1^{(n)} + E^T R_1^{(n)} = 0$, where, by an abuse of notation, E denotes the vector only made up of ones and of the appropriate dimension. Also,

$$\|R\|^2 = R^T R = 2 \|R_1^{(n)}\|^2 = 2.$$

In (3.3.1), the corresponding numerator $R^T L^{(n+1)} R$ is

$$\begin{aligned} & \begin{bmatrix} R_1^{(n)} \\ 0 \\ \vdots \\ 0 \\ R_1^{(n)} \end{bmatrix}^T \begin{bmatrix} L^{(n)} + EE^T & \cdots & L^{(n)} \\ \vdots & \ddots & \vdots \\ L^{(n)} & \cdots & L^{(n)} + EE^T \end{bmatrix} \begin{bmatrix} R_1^{(n)} \\ 0 \\ \vdots \\ 0 \\ R_1^{(n)} \end{bmatrix} \\ &= 2 \left(R_1^{(n)} \right)^T \left(L^{(n)} + EE^T \right) \left(R_1^{(n)} \right) + 2 \left(R_1^{(n)} \right)^T L^{(n)} \left(R_1^{(n)} \right) \\ &= 4 \left(R_1^{(n)} \right)^T L^{(n)} \left(R_1^{(n)} \right) = 4\lambda_1^{(n)}. \end{aligned}$$

Table 1: Numerical evidence shows that both the smallest and the second largest eigenvalues grow at a factorial-like speed.

n	$\lambda_1^{(n)}$	$\lambda_1^{(n+1)}/\lambda_1^{(n)}$	$\lambda_{n!-1}^{(n)}$	$\lambda_{(n+1)!-1}^{(n+1)}/\lambda_{n!-1}^{(n)}$
4	-2	1	6.6055	1
5	-5.0835	2.5417	30.0293	4.5460
6	-20.2413	3.9817	166.1372	5.5324
7	-102.9541	5.0860	1083.7641	6.5233

Thus,

$$\lambda_1^{(n+1)} \leq 2\lambda_1^{(n)}.$$

□

By a very similar method, it can also be proved, as shown next, that the second largest eigenvalue $\lambda_{n!-1}^{(n)}$, which is positive, grows at least exponentially fast.

Proposition 3.3.3. $\lambda_{n!-1}^{(n)} \geq 2^{n-2}\lambda_1^{(2)} = 2^{n-2} > 0$.

Proof. Using the identity

$$\lambda_{(n+1)!-1}^{(n+1)} = \max_{E^T R=0} \frac{R^T L^{(n+1)} R}{R^T R},$$

with a particular choice of

$$R = \begin{bmatrix} R_{n!-1}^{(n)} \\ 0 \\ \vdots \\ 0 \\ R_{n!-1}^{(n)} \end{bmatrix},$$

where $R_{n!-1}^{(n)}$ is the eigenvector associated with the second largest eigenvalue $\lambda_{n!-1}^{(n)}$ of $L^{(n)}$, leads to $\lambda_{(n+1)!-1}^{(n+1)} \geq 2\lambda_{n!-1}^{(n)}$ and thus proves the result. □

The above bounds for $\lambda_1^{(n)}$ and $\lambda_{n!-1}^{(n)}$ are far from tight even as far as their asymptotic orders are concerned. Numerical evidence is collected in the following table:

A reasonable conjecture will be that both the smallest and the second largest eigenvalues grow at a factorial-like speed. More precisely, we believe that

$$\lim_{n \rightarrow +\infty} \frac{\lambda_1^{(n+1)}}{\lambda_1^{(n)}(n-1)} = c_1 \geq 1,$$

and that

$$\lim_{n \rightarrow +\infty} \frac{\lambda_{(n+1)!-1}^{(n+1)}}{\lambda_{n!-1}^{(n)}(n+1/2)} = c_2 \geq 1.$$

3.4 Concluding Remarks

The \sqrt{n} lower-bound conjecture of Bukh and Zhou is still open and seems quite reasonable in view of the fact that $\mathbb{E}LCS(\sigma_1, \sigma_2) \sim 2\sqrt{n}$, in case σ_1 is uniform and σ_2 arbitrary (again, see [17]). We do not have a proof of this conjecture, but let us nevertheless present, next, a quick $\sqrt[3]{n}$ lower bound result.

We start with a lemma describing a balanced property among the lengths of the *LCS*s of pairs of any three arbitrary deterministic permutations. This result is essentially due to Beame and Huynh-Ngoc ([5]).

Lemma 3.4.1. *For any $\pi_i \in \mathcal{S}_n$ ($i = 1, 2, 3$),*

$$LCS(\pi_1, \pi_2)LCS(\pi_2, \pi_3)LCS(\pi_3, \pi_1) \geq n.$$

Proof. The proof of Lemma 5.9 in [5] applies here with slight modification. We further note that this inequality is tight, since letting $\pi_1 = \pi_2 = id$ and $\pi_3 = rev(id)$, which is the reversal of the identity permutation gives, $LCS(\pi_1, \pi_2)LCS(\pi_2, \pi_3)LCS(\pi_3, \pi_1) = n$. \square

In Lemma 3.4.1, taking $(\pi_1, \pi_2) = (id, rev(id))$ gives, for any third permutation π_3 , $LCS(id, \pi_3)LCS(rev(id), \pi_3) \geq n/LCS(id, rev(id)) = n$. But, since $LCS(id, \pi_3)$ and $LCS(rev(id), \pi_3)$ are respectively the lengths of the longest increasing/decreasing subsequences of π_3 , this lemma can be considered to be a generalization of a well-known classical result of Erdős and Szekeres (see [34]).

We are now ready for the cubic root lower bound.

Proposition 3.4.2. *Let P be an arbitrary probability distribution on \mathcal{S}_n and let σ_1 and σ_2 be two i.i.d. random permutations sampled from P . Then, for any $n \geq 1$, $\mathbb{E}_P[LCS(\sigma_1, \sigma_2)] \geq \sqrt[3]{n}$.*

Proof. Let π_1, π_2 and $\pi_3 \in S_n$ and set $L(\pi_i) := \sum_{\pi_1 \in S_n} p(\pi_1) LCS(\pi_1, \pi_i) = \sum_{\pi_1 \in S_n} LCS(\pi_i, \pi_1) p(\pi_1)$, $i = 2, 3$. Then,

$$\begin{aligned} & L(\pi_2) + LCS(\pi_2, \pi_3) + L(\pi_3) \\ &= \sum_{\pi_1 \in S_n} p(\pi_1) (LCS(\pi_1, \pi_2) + LCS(\pi_2, \pi_3) + LCS(\pi_3, \pi_1)) = 3\sqrt[3]{n} \sum_{\pi_1 \in S_n} p(\pi_1) = 3\sqrt[3]{n}, \end{aligned} \tag{3.4.1}$$

by the arithmetic mean-geometric mean inequality and the previous lemma. Further, summing over $p(\pi_2)$ in (3.4.1) gives:

$$\begin{aligned} & \sum_{\pi_2 \in S_n} p(\pi_2) (L(\pi_2) + LCS(\pi_2, \pi_3) + L(\pi_3)) \\ &= \sum_{\pi_2 \in S_n} p(\pi_2) L(\pi_2) + L(\pi_3) + L(\pi_3) \geq 3\sqrt[3]{n}. \end{aligned}$$

Repeating this last procedure but with weights over $p(\pi_3)$ leads to

$$\sum_{\pi_2 \in S_n} p(\pi_2) L(\pi_2) + 2 \sum_{\pi_3 \in S_n} p(\pi_3) L(\pi_3) = 3 \sum_{\pi \in S_n} p(\pi) L(\pi) \geq 3\sqrt[3]{n}. \tag{3.4.2}$$

However,

$$\begin{aligned} \mathbb{E}_P[LCS(\sigma_1, \sigma_2)] &= \sum_{\pi_1 \in S_n} \sum_{\pi_2 \in S_n} p(\pi_1) LCS(\pi_1, \pi_2) p(\pi_2) \\ &= \sum_{\pi_1 \in S_n} p(\pi_1) \sum_{\pi_2 \in S_n} LCS(\pi_1, \pi_2) p(\pi_2) \\ &= \sum_{\pi \in S_n} p(\pi) L(\pi). \end{aligned}$$

Combining this last identity with (3.4.2) proves the result. \square

The above proof is simple; it basically averages out each $LCS(\cdot, \cdot)$ as $\sqrt[3]{n}$ on the summation weighted by P . However, in view of the original conjecture, and our partial results, the cubic root lower-bound is not tight. Apart from our curiosity concerning this $\sqrt[3]{n}$ conjecture, it would be interesting to know the exact asymptotic order of the smallest eigenvalue $\lambda_1^{(n)}$ of $L^{(n)}$. In contrast, the largest eigenvalue $\lambda_n^{(n)}$ corresponding to the uniform distribution is known to be asymptotically of order $2n!\sqrt{n}$, since it is equal to the length of the LISs of a uniform random permutation of $[n]$ scaled by $n!$. In this sense, the study of the length

of the *LCSs* between a pair of i.i.d. random permutations having an arbitrary distribution, or equivalently, the study of $L^{(n)}$, can be viewed as an extension of the study of the length of the *LISs* of a uniform random permutation of $[n]$. Having a complete knowledge of the distribution of all the eigenvalues of $L^{(n)}$ would be a nice achievement.

REFERENCES

- [1] Antonio Auffinger and Michael Damron. Differentiability at the edge of the percolation cone and related results in first-passage percolation. *Probability Theory and Related Fields*, 156(1-2):193–227, 2013.
- [2] Antonio Auffinger and Michael Damron. A simplified proof of the relation between scaling exponents in first-passage percolation. *The Annals of Probability*, 42(3):1197–1211, 2014.
- [3] Antonio Auffinger, Michael Damron, and Jack Hanson. *50 Years of First-Passage Percolation*, volume 68 of *University Lecture Series*. American Mathematical Society, 2017.
- [4] Jinho Baik, Percy Deift, Ken McLaughlin, Peter Miller, and Xin Zhou. Optimal tail estimates for directed last passage site percolation with geometric random variables. *Advances in Theoretical and Mathematical Physics*, 5(6):1–41, 2001.
- [5] Paul Beame and Dang-Trinh Huynh-Ngoc. On the value of multiple read/write streams for approximating frequency moments. In *Foundations of Computer Science, 2008. FOCS'08. IEEE 49th Annual IEEE Symposium on*, pages 499–508. IEEE, 2008.
- [6] Michel Benaïm and Raphaël Rossignol. Exponential concentration for first passage percolation through modified poincaré inequalities. In *Annales de l'IHP Probabilités et statistiques*, volume 44, pages 544–573, 2008.
- [7] Itai Benjamini, Gil Kalai, and Oded Schramm. First passage percolation has sublinear distance variance. *The Annals of Probability*, 31(4):1970–1978, 2003.
- [8] Thierry Bodineau and James Martin. A universality property for last-passage percolation paths close to the axis. *Electron. Comm. Probab*, 10:105–112, 2005.

- [9] Stéphane Boucheron, Gábor Lugosi, and Pascal Massart. *Concentration inequalities: A nonasymptotic theory of independence*. OUP Oxford, 2013.
- [10] Boris Bukh and Lidong Zhou. Twins in words and long common subsequences in permutations. *Israel Journal of Mathematics*, 213(1):183–209, 2016.
- [11] Sourav Chatterjee. The universal relation between scaling exponents in first-passage percolation. *The Annals of Mathematics*, 177(2):663–697, 2013.
- [12] Michael Damron, Jack Hanson, and Philippe Sosoe. Subdiffusive concentration in first-passage percolation. *Electron. J. Probab*, 19(109):1–27, 2014.
- [13] Michael Damron, Jack Hanson, and Philippe Sosoe. Sublinear variance in first-passage percolation for general distributions. *Probability Theory and Related Fields*, 163(1-2):223–258, 2015.
- [14] Richard Durrett and Thomas M Liggett. The shape of the limit set in Richardson’s growth model. *The Annals of Probability*, 9(2):186–193, 1981.
- [15] Ruoting Gong, Christian Houdré, and Jüri Lember. Lower bounds on the generalized central moments of the optimal alignments score of random sequences. *Journal of Theoretical Probability*, pages 1–41, Dec 2017.
- [16] Christian Houdré and Ümit Işlak. A central limit theorem for the length of the longest common subsequences in random words. *arXiv preprint arXiv: 1408.1559v4*, 2017.
- [17] Christian Houdré and Ümit Işlak. A central limit theorem for the length of the longest common subsequence in random words. *arXiv preprint arXiv:1408.1559v4*, 2017.
- [18] Christian Houdré and Jingyong Ma. On the order of the central moments of the length of the longest common subsequences in random words. *High Dimensional Probability: The Cargèse Volume*, Progress in Probability 71, Birkhäuser:105–137, 2016.
- [19] Christian Houdré and Heinrich Matzinger. Closeness to the diagonal for longest common subsequences. *Electronic Communications in Probability*, 21(2):1–19, 2016.

- [20] Christian Houdré and Chen Xu. Concentration of geodesics in Bernoulli directed last passage percolation. *In Preparation*, 2016.
- [21] W. Jockusch, J. Propp, and P. Shor. Random Domino Tilings and the Arctic Circle Theorem. *arXiv preprint arXiv: math/9801068*, January 1998.
- [22] Kurt Johansson. Shape fluctuations and random matrices. *Communications in Mathematical Physics*, 209(2):437–476, 2000.
- [23] Kurt Johansson. Transversal fluctuations for increasing subsequences on the plane. *Probability Theory and Related Fields*, 116(4):445–456, 2000.
- [24] Harry Kesten. Aspects of first passage percolation. In *École d’Été de Probabilités de Saint Flour XIV-1984*, pages 125–264. Springer, 1986.
- [25] Harry Kesten. On the speed of convergence in first-passage percolation. *The Annals of Applied Probability*, 3(2):296–338, 1993.
- [26] Joachim Krug and Herbert Spohn. Kinetic roughening of growing surfaces. In C. Godreche, editor, *Solids far from equilibrium*, pages 479–582. Cambridge University Press, Cambridge, 1991.
- [27] Jüri Lember and Heinrich Matzinger. Standard deviation of the longest common subsequence. *The Annals of Probability*, 37(3):1192–1235, 2009.
- [28] R Marchand. Strict inequalities for the time constant in first passage percolation. *The Annals of Applied Probability*, 12(3):1001–1038, 2002.
- [29] James B. Martin. Limiting shape for directed percolation models. *The Annals of Probability*, 32(4):2908–2937, 2004.
- [30] Charles M. Newman and Marcelo S.T. Piza. Divergence of shape fluctuations in two dimensions. *The Annals of Probability*, 23(3):977–1005, 1995.
- [31] Robin Pemantle and Yuval Peres. Planar first-passage percolation times are not tight. In *Probability and phase transition*, pages 261–264. Springer, 1994.

- [32] Shaiy Pilpel. Descending subsequences of random permutations. *Journal of Combinatorial Theory, Series A*, 53(1):96–116, 1990.
- [33] WanSoo T. Rhee. On rates of convergence for common subsequences and first passage time. *The Annals of Applied Probability*, 5(1):44–48, 1995.
- [34] Dan Romik. *The surprising mathematics of longest increasing subsequences*, volume 4. Cambridge University Press, 2015.
- [35] Hermann Rost. Non-equilibrium behaviour of a many particle process: density profile and local equilibria. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 58(1):41–53, 1981.
- [36] Timo Seppäläinen. Hydrodynamic scaling, convex duality, and asymptotic shapes of growth models. *Markov Processes and Related Fields*, 4(1):1–26, 1998.
- [37] Yu Zhang. Shape fluctuations are different in different directions. *The Annals of Probability*, 36(1):331–362, 2008.
- [38] Yu Zhang. On the concentration and the convergence rate with a moment condition in first passage percolation. *Stochastic Processes and their Applications*, 120(7):1317–1341, 2010.